A Standard Course, and Lecture Notes in

**Analytical Mechanics**

The Course Presented by

Ph.D Professor Dr., and Head of the Mathematics Department

Ibrahim Fahmy Mikhail

in

Lagrangian and Hamiltonian Mechanics

with Applications to

Classical, Statistical, Relativistic and Quantum Mechanics

Presenting the Hamilton Equations for

Holonomic Conservative Systems

and other problems with solutions

And the

Theory of Small Oscillations about the Position of Stable Equilibrium
Examples: Given \( q = (kx/(x^2+y^2), 0, -kz/(x^2+y^2)) \), show that the motion is irrotational and the fluid is incompressible. Find \( \Phi \) (potential velocity).

1. \( \nabla \times q = 0 \) leads to \( \Phi = -k \log r \).

2. \( \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \) is Laplace's equation.

3. \( \frac{1}{2} \cdot q = \frac{k^2 (x^2+y^2)}{(x^2+y^2)^2} \)
Motion of a system of particles on a rigid body in space.

For a system of particles, the motion is governed by the equations of motion.

Linear motion: \[ \ddot{r} = \mathbf{F} / \mathbf{m} \quad \text{or} \quad \mathbf{F} = \mathbf{m} \ddot{r} \]

Rotational motion about a fixed point \( O \) is taken as origin.

\[ \mathbf{\omega} = \mathbf{\omega} \times \mathbf{r} \]

Angular momentum: \[ \mathbf{L} = \mathbf{r} \times \mathbf{p} \]

Angular momentum about a fixed point \( O \) is given by \[ \mathbf{L} = \mathbf{m} \ddot{r} + \mathbf{r} \times \mathbf{F} \]

Angular momentum of a mass \( m \) at \( G \) about \( O \):

\[ \mathbf{L} = \sum m_i \mathbf{r}_i \times \mathbf{v}_i \]

Equation of motion:

\[ \mathbf{F} = \mathbf{m} \ddot{r} + \mathbf{r} \times \mathbf{F} = \mathbf{N} + \mathbf{r} \times \mathbf{M} \]

No external forces are acting on the system, so the net external force and moment are zero.
\[ h_0 = h_G + \sum \vec{r} \times \vec{M} \]  

\[ h_1 = h_G + \sum \vec{r} \times \vec{M} \]  

\[ \frac{dh}{dt} = h_G + \vec{v} \times \vec{r} \]  

\[ h_2 = h_G + \vec{v} \times \vec{r} + \frac{1}{2} \vec{a} \times \vec{r} \]  

\[ \frac{dh}{dt} = h_G + \vec{v} \times \vec{r} + \frac{1}{2} \vec{a} \times \vec{r} \]  

Rate of change of angular mom. about the center of gravity \( G \).  

Moment of external forces about \( G \).  

- The center of gravity \( G \) can be treated as a fixed point.  
- For the rotational motion about any point \( P \) (not necessarily fixed).  

Moreover, from est. IV, \( IV \) we find:  

\[ N_0 = h_G + \sum \vec{r} \times \vec{M} \]  

\[ \frac{dN}{dt} = \frac{\partial N}{\partial \vec{N}} \]  

\[ \frac{dN}{dt} = \vec{v} \times \vec{r} + \frac{1}{2} \vec{a} \times \vec{r} \]  

\[ S_i = \frac{\partial N}{\partial \vec{M}} \]  

\[ 9 \]  

\[ S_i = \vec{r} \times \vec{m} \]  

\[ \sum S_i \times \vec{m} = \sum (\vec{r} \times \vec{m}) \]  

\[ \sum \vec{r} \times \vec{m} = \vec{r} \times \sum \vec{m} \]  

\[ \vec{h}_P = \sum \vec{r} \times \vec{m} \]  

\[ \vec{h}_P = \vec{h}_G + \vec{v} \times \vec{r} + \frac{1}{2} \vec{a} \times \vec{r} \]  

\[ \vec{h}_P = \vec{h}_G + \vec{v} \times \vec{r} + \frac{1}{2} \vec{a} \times \vec{r} \]  

Which is a generalization of \((IV)\).
\[ \mathbf{N}_p = \sum \mathbf{S}_i \times \mathbf{F}_{ii} = \sum (\mathbf{e}_i + \mathbf{q}) \times \mathbf{F}_{ii} \]

\[ T \mathbf{e}_i \times \mathbf{F}_{ii} + q \times T \mathbf{e}_i = \mathbf{F} \times \mathbf{N}_i \]

\[ \mathbf{N}_p = \mathbf{h}_q + q \times \mathbf{M}_i^e \]

Generalization of eqn. \( I \) is the moment of external forces about any point \( P \).

Rule of Change of Moment about \( G \) + Moment of the acceleration of \( G \) about \( P \).

+ Taking the deri.

Moreover from eqn. \( II \), we have \[ \dot{\mathbf{N}}_p = \mathbf{h}_G + q \times \mathbf{M}_i^e + \dot{q} \times \mathbf{N}^e \]

\( \dot{q} \) = velocity of \( G \) relative to \( P \).

From \( V \)

\[ \mathbf{N}_p = \mathbf{N}_p + \dot{q} \times \mathbf{M}_i^e \]

\[ (\mathbf{v}_G - \mathbf{v}_p) \times \mathbf{M}_i^e = \mathbf{N}_p - \mathbf{v}_p \times \mathbf{M}_i^e \]

Which is in accordance with \( I, IV \)

From eqn. \( V \), we conclude that \[ \dot{\mathbf{N}}_p = \mathbf{N}_p \] in the following cases:

(i) \( P \) is fixed \( \Rightarrow \mathbf{v}_p = 0 \) \( \Rightarrow \) eqn. \( II \)

(ii) \( P \) coincides with \( G \) \( \Rightarrow \mathbf{v}_p = \mathbf{v}_G \) \( \Rightarrow \) eqn. \( IV \)

(iii) \( \mathbf{v}_p \parallel \mathbf{v}_G \)

(iv) \( \mathbf{v}_G = 0 \) Pure rotational motion about \( G \).
\[ \omega_0 = \int_0^t \omega(t) \, dt \]

Let \( \omega \) be the angular velocity of the body at any point of the body \( \mathbf{r} \), rotates (moves) about the surface of a sphere of center \( \mathbf{0} \) with radius \( R \) and angular velocity \( \omega \).

\[ \omega = \omega \times \mathbf{r} \quad \text{the velocity of a point moving on a sphere.} \]

Therefore, \( \omega_0 = \omega \times \mathbf{r}_0 \Rightarrow \mathbf{h}_0 = \mathbf{r}_0 \times m_0 \omega \times \mathbf{r}_0 \]

\[ \mathbf{h}_0 = \sum m_i \mathbf{r}_i = \sum m_i \left( \mathbf{r}_i \times \omega \times \mathbf{r}_0 \right) \]

\[ \mathbf{h}_0 = \sum m_i \left( \mathbf{r}_i \times \omega \times \mathbf{r}_0 \right) = \sum m_i \left( \mathbf{r}_i \times \mathbf{r}_0 \times \omega \right) \]

\[ \mathbf{h}_0 = \sum m_i \mathbf{r}_i \times \mathbf{r}_0 \times \omega \]
\[
\begin{align*}
\mathbf{h}_0 & = \mathbf{w}_x - \mathbf{w}_y \mathbf{F} - \mathbf{w}_z \mathbf{E} \\
\mathbf{h}_1 & = \mathbf{w}_x \mathbf{F} + \mathbf{w}_y \mathbf{B} - \mathbf{w}_z \mathbf{D} \\
\mathbf{h}_2 & = -\mathbf{w}_x \mathbf{E} - \mathbf{w}_y \mathbf{D} + \mathbf{w}_z \mathbf{C}
\end{align*}
\]

Since the plane \( \mathbf{h}_0 \sim \mathbf{I}_W \sim \mathbf{I}_0 \), \( \mathbf{h}_1 \) is a generalization

\( \mathbf{0} \) is a fixed point of a rigid body.

\[
\begin{bmatrix}
\mathbf{h}_{0x} \\
\mathbf{h}_{0y} \\
\mathbf{h}_{0z}
\end{bmatrix} = \begin{bmatrix}
\mathbf{A} & -\mathbf{F} & -\mathbf{E} \\
-\mathbf{F} & \mathbf{B} & -\mathbf{D} \\
-\mathbf{E} & -\mathbf{D} & \mathbf{C}
\end{bmatrix} \begin{bmatrix}
\mathbf{w}_x \\
\mathbf{w}_y \\
\mathbf{w}_z
\end{bmatrix}
\]

The matrix \( \mathbf{H} \) is called the inertia matrix at the point \( \mathbf{0} \).

* Which is a tensor of a 2nd order because matrices in general are tensors of 2nd order. \( \mathbf{h}_0 = \mathbf{I}_0 \mathbf{\omega} \)
Remarks:

(i) The following notations are used for moment and products of inertia:

(ii) If \( OX, OY, OZ \) are chosen to be the principle axes of \( O \),

then \( I_{xy} = 0, I_{zx} = 0 \) for \( xO \) to be a principle axis.

In general, \( OX, OY, OZ \) be the principle axes of \( O \),

products of inertia \( T \) are 2-axes.

Accordingly, \( h_0 = \mathbf{A} \mathbf{n} \mathbf{i} + \mathbf{B} \mathbf{y} \mathbf{j} + \mathbf{C} \mathbf{z} \mathbf{k} \)

(iii) For a plane lamina moving in the plane \( XY \) We have:

\[ a_x = b_y = 0 \]

\( Q \) we be a principle axis and hence

\[ I_{xy} = 0 \]

\[ D = E = 0 \]

- Consequently, \( h_{0x} = h_{0y} = 0 \) \( \rightarrow h_{0z} = C \mathbf{u}_2 = C \mathbf{w} \)

Similar to the well known relation \( I = I_0 + \mathbf{I}_0 \)

+ where \( I \) is the moment of inertia about an axis \( (OZ) L \) to the

plane of motion and passing through the fixed point \( O \).

(iv) For the general motion of a rigid body \( \mathbf{r} \) is \( (\mathbf{r}) \) fixed point,

we have \( \mathbf{b}_G = \mathbf{I} \mathbf{e}_i \mathbf{x} m_i \mathbf{r}_i \)

\[ \mathbf{L} \mathbf{e}_i \mathbf{x} m_i (\mathbf{v}_i + \mathbf{r}) = \]

\[ \mathbf{L} \mathbf{e}_i \mathbf{x} m_i (\mathbf{v}_i + \mathbf{r}) = \]

\[ \mathbf{m}_i (\mathbf{e}_i \mathbf{x} \mathbf{e}_i) \mathbf{e}_i + \mathbf{I} \mathbf{m}_i \mathbf{r} \]

\[ \rightarrow \mathbf{0} \]
But from the laws of the Center of Gravity

\[ L_{m_i} e_i = 0, \quad \sum L_{m_i} \dot{e}_i = 0, \quad L_{m_i} \dot{e}_i = 0 \]

or

\[ \sum L_{m_i} e_i = 0 \]

but \( \dot{e}_i \) represents the velocity of \( p_i \) relative to \( G \).

In this relative motion \( G \) is considered to be fixed and thus \( p_i \) moves relative to \( G \) on a surface of a sphere of north center \( G \), radius \( \frac{GM_i}{\omega^2} \), which is fixed in my.

and angular velocity \( \omega \) of the sphere body.

Consequently \( \omega \times \dot{e_i} = \dot{e}_i \)

\[ L_{m_i} e_i \times (\omega \times e_i) = L_{m_i} e_i^2 \omega - \omega (\omega \cdot e_i) e_i \]

\[ e_i = \omega (\omega \cdot e_i) + \omega \times e_i \]

Therefore

\[
\begin{bmatrix}
\mathbf{w}_x \\
\mathbf{w}_y \\
\mathbf{w}_z
\end{bmatrix}
= \begin{bmatrix}
A' & -E' & -F' \\
F' & B' & -D' \\
-E' & D' & C'
\end{bmatrix}
\begin{bmatrix}
\omega x \\
\omega y \\
\omega z
\end{bmatrix}
\]

\[ \mathbf{h}_g = \mathbf{I} \mathbf{w} \]

As regards rotational motion \( G \) can be treated as a fixed point.

The angular momentum about any point \( P \) can be obtained by using equation (III)

\[ \mathbf{h}_g = \mathbf{h}_g + \mathbf{g} \times \mathbf{m} + \mathbf{u} \]

Position of \( G \) relative to \( P \)
taking \( g = \mathbf{u} \triangleq \mathbf{u} \times \mathbf{j} \)
as regard a fixed Point $O$, Eq. 1. The Centre of Mass is taken $C$ and the result can be expressed in the same form as Eq. (1) by taking $X = \mathbb{C} \times X$ and using the theorem parallel axes.

The Kinetic Energy of a rigid body in which a point $O$ is fixed is

\[ K.E. = T = \frac{1}{2} I \omega^2 \]

If $O$ is the fixed point of the body then $\omega = \dot{X} = \omega \times X$

Therefore

\[ T = \sum m_i \dot{X}_i \cdot (\omega \times X) = \frac{1}{2} \sum m_i \dot{X}_i \times m_i X_i - \frac{1}{2} \omega \cdot h_0^2 \]

\[ = \frac{1}{2} \left[ w_1 h_{1x} + w_2 h_{2x} + w_2 h_{2y} \right] \approx \text{from III} \]

\[ = \frac{1}{2} \left[ w_1^2 A + w_2^2 B + w_2^2 C \right] - 2 w_1 w_2 F - 2 w_1 w_2 \bar{E} \]

\[ = 2 w_1 w_2 \bar{D} \]

\[ T = \frac{1}{2} \omega^T \bar{h}_0 = \frac{1}{2} \omega^T \bar{I}_0 \bar{w} \]

\[ = \frac{1}{2} \left( w_1, w_2, w_3 \right) \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \]

Since

\[ T = \frac{1}{2} I \omega^2 \]

\[ \frac{1}{N+2} \frac{1}{L^2} \]
Notes

(i) If \( OX, OY, OZ \) are chosen to be the principal axes at \( O \), then all products of inertia i.e. \( 2 = E = F = 0 \) and hence

\[
T = \frac{1}{2} \left[ w_x A + w_y B + w_z C \right]
\]

and we always desire to take the principal axes.

(ii) For a plane laminar laminar moving in the plane \( XY \) we have \( w_x = w_y = 0 \) and

\[
2 = E = F = 0
\]

\[
T = \frac{1}{2} C w_z^2 \]

in agreement with the stated well known result

\[
2 = \frac{1}{2} I w^2 = \frac{1}{2} I \dot{O}^2
\]

(iii) For the general motion of a rigid body

\[
T = \frac{1}{2} \sum m_i \dot{e}_i \cdot \dot{e}_i = \frac{1}{2} \sum m_i (\dot{e}_i + \ddot{r}) \cdot (\dot{e}_i + \ddot{r}) = \frac{1}{2} \sum m_i \dot{e}_i \cdot \dot{e}_i + \frac{1}{2} \sum m_i \dddot{r} \cdot \dot{e}_i + \frac{1}{2} \sum \dot{r} \cdot \dddot{r}
\]

since \( \dot{e}_i \perp \dot{r} \):

\[
T = \frac{1}{2} \omega_1 \omega_1 + \frac{1}{2} M \dot{V}^2
\]

\[
\frac{1}{2} \omega_1 \dot{I} \frac{\omega_1}{\dot{g}} + \frac{1}{2} M \dot{V}^2
\]

\[
E_k \text{ due to rotational motion about } \dot{g}, \quad \dot{\omega}
\]

\[
\text{IX motion due to } \dot{g}
\]
8. Some general remarks.

(i) If rotating axes are used, then for any vector $\mathbf{V}$,

$$\frac{d\mathbf{V}}{dt} = \frac{d\mathbf{V}}{dt} + \mathbf{\Omega} \times \mathbf{V}$$

hence if $\mathbf{V}_0 = \mathbf{V}(t=0)$

(Diffusion, assuming that the axes are fixed.)
Electro Magnetism.

Basic Electrostatic Concepts and Laws:

- Coulomb's Law

There are two kinds of interactions of elementary particles which are noticeable even at large distances, electrical forces, proportional to the charge.

- Magnetic forces which are proportional to the product of the charge and the velocity of the particle.

In electrostatics only forces of the 1st kind exist.

Two particles with the electric charges $Q_1$, $Q_2$ at a distance $r$ from each other interact with a force equal to $K Q_1 Q_2 \frac{1}{r^2}$, where $K$ is a Constant of proportionality. In order to take into consideration also the direction of the force $F$ the distance vector into direction from $Q_1$ to $Q_2$ is designated by $x$.

Then the force which acts on the charge $Q$ will be expressed by $F = \frac{Q Q_2}{4 \pi \varepsilon_0 r^2}$, both in magnitude and direction.

This relation implies that there is repulsion and attraction if the charges have like signs and unlike signs, respectively.

- The Constant of proportionality $K$ depends on the units used.
- Put $K = \frac{1}{4 \pi \varepsilon_0}$; $\varepsilon_0$ becomes a constant natural constant which acts as the Constant of the System of units.

Electric Field Strength and Potential; Eqts. of force and work:

Consider a number of charges $q_1$, $q_2$, etc., at the

fixed points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, and a moving charge with the coordinate $(x, y, z)$, the force $F$ acting on $q$ is then the vector sum of all the forces originating from the separate $q_i$'s.
A. Mech.

Let \( x, y, z \) be rotating vectors with angular velocities \( \frac{\partial \mathbf{V}}{\partial t} = \frac{\partial \mathbf{V}_1}{\partial t} + \frac{\partial \mathbf{V}_2}{\partial t} + \frac{\partial \mathbf{V}_3}{\partial t} \).

Then \( \mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 \).

\[
\frac{d\mathbf{V}}{dt} = \left( \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} \right) \mathbf{V} = \mathbf{V}_1 \mathbf{d} \mathbf{i} + \mathbf{V}_2 \mathbf{d} \mathbf{j} + \mathbf{V}_3 \mathbf{d} \mathbf{k}
\]

The unit vector \( \mathbf{i} \) can be regarded as the position vector of a point moving on the surface of a sphere of center \( O \), fixed point, and radius equal to unity, with an angular velocity \( \omega \). Therefore, \( \frac{d}{dt} \) represent the velocity of this point and thus given by

\[
\frac{d\mathbf{V}}{dt} = \frac{d\mathbf{V}_1}{dt} + \omega \times \mathbf{V}
\]

The laws of motion obtained before must then be modified to include the terms which arise due to the rotation of the axes when performing any differentiation operation.

Consequently, in linear motion \( F = \mathbf{F}_g \).

\[
\mathbf{F}_g = -\mathbf{F}_0
\]

\[
\frac{d\mathbf{V}}{dt} = -\frac{\mathbf{d} \mathbf{V}_0}{dt} + \omega \times \mathbf{V}_0
\]

(2) (in rotating case)
\[ N_p = \dot{\theta}_p + \frac{2}{3} \times \dot{\theta}_g + 9 \times \omega \times M \dot{\theta}_g \]

The acceleration whose components are taken relative to the rotation axis is

\[ N_p = \dot{\theta}_p + \frac{2}{3} \times \dot{\theta}_g + 9 \times \omega \times M \dot{\theta}_g \]

Remarks: Some Rules to determine the principal axes of inertia.

(a) The axis perpendicular to a plane lamina is a principal axis of inertia at the point of intersection with the plane of the lamina.

\[ I_{yz} = \sum m z^2 c_{20} \quad I_{zx} = \sum m z^2 c_{12} \]

(b) The axis of symmetry is a principal axis at all of its points.

\[ I_{xy} = I_{yx} \]
\[ I_{y2} = dm \cdot z^2 + \int dm(\alpha_z^2) = 0 \]

similarly

\[ I_{z2} = 0 \]

(c) The axis perpendicular to a plane of symmetry is a principal axis.

the point of intersection with the plane of symmetry.

\[ d I_{y2} = dm \cdot z^2 + \int dm(\alpha_x^2) \]

There for all 2 is a principal axis at the point 0.

(d) The principal axis which passes through the center of gravity is a principal axis at all of its points.

Let \( O2 \) be a p. A at the point 0 and passes through the center of gravity 0.

\[ I_{y2} = I_{y2} - M(OX) = \]

\[ I_{z2} = I_{z2} - M(OY) = \]

\[ I_{x2} = I_{x2} - M(OZ) = 0 \]

\[ I_{y2} = I_{y2} + M(0X) = I_{y2} = 0 \]

\[ I_{z2} = I_{z2} + M(0Y) = I_{z2} = 0 \]

\[ I_{x2} = I_{x2} + M(0Z) = I_{x2} = 0 \]

\[ c^2 \text{ is a principal axis at any point } O' \]
The inertia matrix \[ \mathbf{I} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix} \]

The eigenvalues of the inertia matrix \( \mathbf{I} \) are \( \lambda_1, \lambda_2, \lambda_3 \), and the corresponding eigenvector \( \mathbf{v} \) is \( \mathbf{v} = [x_1, x_2, x_3]^T \).

\[ \begin{bmatrix} \lambda_1 \end{bmatrix} \]

It is well known that the rotational motion is determined in a plane by one angle.

\[ \mathbf{w} = \dot{\mathbf{v}} \] is the angular velocity of the luminance, \( \mathbf{w} \) is a unit vector along the direction to the plane of motion.

\[ \Theta = \dot{\phi} + \alpha ; \phi = \beta + \phi \]

\[ \alpha = \theta - \alpha + \phi \]

\[ \Theta = \alpha \]

In 3-D, however, we need 3 angles (Eulerian Angles) to determine the angular velocity of the body. The E-A are defined in the following way.

Let \( x, y, z \) be fixed axes in the space.

Let \( x', y', z' \) be axes which are fixed in the body and rotating with it.

\[ \theta, \phi, \psi \]

\[ \begin{bmatrix} \Theta \\ \alpha \\ \phi \end{bmatrix} \]

\[ \mathbf{E} \]
The E.A. is the angle $\theta$ between the axis $Z$ fixed in the body and $Z$ (fixed in the space) which

$\circ$ $Z$ is the meridian plane (in magnetic field) i.e.
plane $(Z, Z)$ is usually called the meridian plane.

The angle $\phi$ between the meridian plane $(Z, Z)$ and the plane $(X, Z)$ fixed in the space.

$\Psi$ is the angle the M.P. (plane $(Z, Z)$) and the plane $(Z, X)$ which lies in the body.

Plane $Z$ contains the axes $(X, Y, X')$
$P \perp Z = (x, x', x'')$
$\bigcirc X'$ is the intersection of the M.P and plane $Z$

$\bigcirc X'' = M.P. \perp Z$

Plane $Z \perp (x, x, x')$, $Y'$

Therefore $\psi \perp Z$, $Z$ is the M.P $(Z, Z, X, X')$

Sometimes called the line of nodes

The axes $(X, Y, Z)$ can be related to the axes $(x, x, x')$ in the following way:

(a) Rotation about $\phi$ by the angle $\phi$
(b) Rotation about $Y'$ by an angle $\theta$

Rotation in the M.P. according to this rotation $A$ to this $B$

$Z \rightarrow Z', \quad X' \rightarrow X''$

(c) Rotation about $Z$ by the angle $\phi$ (with this $R$)

$X'' \rightarrow X', \quad Y' \rightarrow Y$
The angular velocity of the body \( \omega = \dot{\theta} + \phi + \psi \)

The components of the angular velocity along the axes \( X, Y, Z \) fixed in the space.

Let \( I, J, K \) be unit vectors along the fixed axes \( X, Y, Z \), then \( \dot{\phi} = \dot{\phi} K \) (4)

\[
\dot{\theta} = \dot{\theta} (-\sin \theta I + \cos \theta J) \quad (2)
\]

\[
\dot{\psi} = \dot{\psi} \sin \theta \cos \phi I + \dot{\psi} \sin \theta \sin \phi J + \dot{\psi} \cos \theta K \quad (3) \Rightarrow \\
\dot{\psi} = \dot{\psi} \left( \sin \theta \cos \phi I + \sin \theta \sin \phi J + \cos \theta K \right) \\
\dot{\psi} = \dot{\psi} (\sin \theta \cos \phi I + \sin \theta \sin \phi J + \cos \theta K) \\
\dot{\psi} = \dot{\psi} \left( -\sin \theta \sin \phi + \psi \sin \theta \cos \phi \right) I + \left( \dot{\theta} \cos \phi + \psi \sin \theta \sin \phi \right) J + \left( \dot{\phi} \cos \phi \right. \sin \theta \cos \theta \left. \right) K
\]
The Components of the A. velocity along the rotating axes 
\( x, \ y, \ z \), let \( i, \ j, \ k \) be unit vectors along 
the axes \( x, \ y, \ z \), then

\[
\dot{\phi} = \dot{\phi} \times \hat{\omega} \\
\dot{\theta} = \dot{\theta} (\sin \phi \ i + \cos \phi \ j) \quad (2)
\]

\[
\hat{\omega} = \dot{\phi} \left( -\sin \theta \cos \phi \ i + \sin \theta \sin \phi \ j + \cos \theta \ k \right)
\]

Therefore

\[
\hat{\omega} = (\dot{\theta} \sin \phi - \dot{\phi} \sin \theta \cos \phi) i + (\dot{\theta} \cos \phi + \dot{\phi} \sin \theta \sin \phi) j \\
+ (\dot{\phi} + \dot{\phi} \cos \theta) k
\]

Note: The velocity of rotation can also be determined in the same way, using the rotating axes.

For example, in the Case of spherical polar coordinates, the axes \( (x', y', z') \) are taken to be \( (x'', y'', z) \),

where here \( \dot{\phi} = 0 \)

hence \( \dot{\theta} = \dot{\theta} j \)

\[
\dot{\omega} = -\dot{\phi} \sin \theta \ i + \dot{\theta} j + \dot{\phi} \cos \theta \ k
\]

\[
\dot{\omega} = -\dot{\phi} \sin \theta \ i + \dot{\theta} j + \dot{\phi} \cos \theta \ k
\]
Applications:

1) Euler's Equations:
   \( \dot{N}_0 = N_0 \)
   
   - The rotational motion of a body about a fixed point \( O \) for any fixed
     point \( O \), relative to a rotating frame of reference or a fixed axes.
     
   is given by \( N_0 = \dot{N}_0 + \dot{\Omega} \times \dot{N}_0 \)  \( \left( \frac{d}{dt} \right) \) (assuming fixed axes)
   
   \( \dot{\Omega} = \omega \), velocity of the axes
   
   - If we further assume that the rotating axes are
     fixed in the body, then \( \dot{\Omega} = \omega \), velocity of the body
   
   Moreover, it will be assumed that \( O \) is a fixed point of the
   body, \( \begin{bmatrix} \dot{N}_0x \\ \dot{N}_0y \\ \dot{N}_0z \end{bmatrix} = \begin{bmatrix} A \\ D \\ E \end{bmatrix} \begin{bmatrix} \dot{\Omega}x \\ \dot{\Omega}y \\ \dot{\Omega}z \end{bmatrix} \) and the rotating axes
   \( \begin{bmatrix} \dot{N}_0x \\ \dot{N}_0y \\ \dot{N}_0z \end{bmatrix} = \begin{bmatrix} \dot{A} \\ \dot{D} \\ \dot{E} \end{bmatrix} \) where the principal axes of inertia
   \( \begin{bmatrix} I_x, I_y, I_z \end{bmatrix} \)
   
   \( L_0 \), accordingly, \( \dot{L}_0 = A \omega_i + B \omega_j + C \omega_k \)  \( (3) \)
   
   Substituting from (2)  \( (3) \) in (4), we find that
   
   \( A \omega_i + B \omega_j + C \omega_k + \begin{bmatrix} i \\ j \\ k \end{bmatrix} \begin{bmatrix} \dot{A} \\ \dot{B} \\ \dot{C} \end{bmatrix} = \frac{N_0}{I} \)
   
   \( A \omega_i - \omega_i \omega_j (B - C) = \frac{N_0}{I} \); \( B \omega_j - \omega_j \omega_i (C - A) = \frac{N_0}{I} \)

   \( C \omega_k - \omega_k \omega_i (A - B) = \frac{N_0}{I} \)  \( (4) \)
Condition for $C$, Eq. 2, to hold:

1. $\mathbf{O}$ is a fixed point of the body.
2. The rotating axes are along the principal axes of inertia of the body at $\mathbf{O}$.

Hence, Equations (12) are called Euler's Equations.

If we further assume that, the moment of external forces about $\mathbf{O}$ is equal to zero $N_0 = 0$ or the body moves under no forces, then Eq. 2 (4) take the form

$$\mathbf{u} = 0, \quad \mathbf{u}_i = 0, \quad \mathbf{u}_j = 0$$

--Note! For a plane lamina, if the $Z$ axis is taken perpendicular to the plane of the lamina, then $x, y$ are the other two principal axes and $z$ is at the point $O$ in the plane of the lamina.

$$I_2 = I_x + I_y = I_m (y^2 + z^2) \text{ and } I_m y^2 + I_m z^2$$

$$C = -2A + B \quad \therefore$$

$$\mathbf{u} = 0 = \dot{w}_1 + \dot{w}_2 \dot{w}_3 = 0 \quad \therefore$$

$$\mathbf{u}_2 = 0 = \dot{w}_2 - \dot{w}_3 \dot{w}_1 = 0 \quad \therefore$$

Taking from 1, 2 \Rightarrow $w_2 = w_1 w_3, $w_1 w_2 = 0 \Rightarrow\text{ integrating } \dot{w}_1 \text{ to } t$

$$\frac{1}{2} (w_1^2 + w_2^2) = \text{ Constant} \quad \therefore$$

$$w_1^2 + w_2^2 = \text{ Constant} \quad \text{For a plane lamina, which is determined from initial Conditions}$$
Example: A, B, C, D is a plane lamina of mass 3m, and in which BC = 2AB, two particles each of mass m are fixed at the mid points of DA and CB, the lamina is free to rotate about its center O which is fixed; the rectangular axes Ox, Oy, Oz are chosen parallel to AB, DA and L to the plane of the lamina, initially the lamina is set in motion with an angular velocity which is along Oy is \( \omega = \frac{T}{2} \), and bind the other two components.

- We first note that
  1. The center O of the lamina is fixed.
  2. Ox, Oy are symmetrical axes passing through O and are thus principal axes of inertia at C, also OZ = Ox, Oy into the plane of the lamina and is accordingly a principal axis.

2. No = \( x^2 + \frac{y^2}{\frac{1}{2}} \) = 0

- We can now apply Euler's Eq 4.

\[
A\omega = -A\omega_1, \quad (B - C) = 0
\]

\[
B\omega_2 = -B\omega_1, \quad (C - A) = 0
\]

\[
C\omega_3 = -C\omega_1, \quad (A - B) = 0
\]

\[
\omega_1 = \frac{A\omega_1}{2}, \quad BC = 2AB = 6a
\]

Moment of inertia of a rectangle is \( \frac{1}{2} \) \( M \) \( b^2 \)

\[
A_1 I_x = \frac{1}{3} \times 2m \left( \frac{a}{2} \right)^2 = \frac{1}{6} m^2
\]

\[
B_1 I_y = \frac{1}{2} \times 3m \left( \frac{a}{2} \right)^2 + \frac{1}{2} m^2 + \frac{1}{2} m^2 = 3m^2
\]

For a plane lamina

\[
C = I_2 = I_x + I_y
\]

Thus, \( A_1 B_1 C = 4m^2 + 3m^2 + 12m^2 = 19m^2 \).
Substituting in equation (1.8), we find
\[ w_1 - w_2 w_3 (2 - 5) = 0 \]
For any plane lamina \( W \), we have
\[ 2 (w_3) - w_2 w_1 (3 - 1) = 0 \]
\[ 2w_3^2 - 2 w_2 w_1 = 0 \]
\[ w_1 = w_2 = w_3 = 0 \] (2a).

Initially we have 3 components at \( t = 0 \), we have 2 components.

In order to express \( w_3 \), intercept of \( w_3 \), we use the following way:
\[ w_3 x (2 \omega_0) + w_2 x (2 \omega_0) = w_2 w_1 + x_1 w_3 = 0 \] (3)

Integrating (3) \( w_1 \to t - \omega_0 \) (From the initial condition)
\[ \frac{1}{2} \omega_1^2 = - \frac{1}{2} \omega_2^2 + C_1 \] (4), at \( t = 0 \)
\[ w_2 = 0 \Rightarrow C_1 = -\frac{3}{2} \omega_2^2 \]
\[ \frac{1}{2} \omega_1^2 = - \frac{1}{2} \omega_2^2 + \frac{3}{2} \omega_2^2 \]
\[ \omega_1 = \sqrt{\omega_2^2 - \omega_3^2} \] (5).

Note \[ |\omega_1| < (\sqrt{3} \omega_2) \] is impossible.

Also \( w_3 \) can be expressed in terms of \( w_2 \) as follows:
\[ w_3 (2 \omega_3) + w_2 (2 \omega_1) = w_2 w_1 + 3 w_2 w_3 = 0 \] (6b)
\[ \Rightarrow \frac{1}{2} w_2^2 + \frac{3}{2} w_3^2 = C_2 \]
\[ \Rightarrow C_2 = \frac{3}{2} \omega_2^2 \]
\[ w_2^2 + 3 w_3^2 = 3 \omega_2^2 \]
\[ 3 \left( \omega_2^2 - \omega_3^2 \right) \Rightarrow |w_3| < \sqrt{3} \omega_2 \] (6c)

We now substitute (5), (6) in (2a) and find that
\[ w_1 = \frac{1}{\sqrt{3}} (3 \omega_2^2 - w_1^2) \quad \text{or} \quad \int \frac{dt}{\sqrt{3} \omega_2^2 - \omega_1^2} = \int \frac{\omega_3^2}{\omega_3^2} dt \]

\[ \omega_1 = \frac{1}{\sqrt{3}} \left( 3 \omega_2^2 - w_1^2 \right) \quad \text{or} \quad \frac{1}{\sqrt{3}} \int \frac{dt}{\omega_1^2 - \omega_2^2} \]
\[
\int \frac{dx}{\sqrt{1-x^2}} = \frac{1}{2} \ln \left( \frac{1+\sqrt{1-x^2}}{1-x} \right)
\]

Notice \( C = \frac{1}{a} \ln \frac{2a}{a-x} + C \)

\[
\int \frac{dx}{a^2 + x^2} = \frac{1}{2a} \tan^{-1} \frac{x}{a} + C
\]

\[
\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \frac{x+a}{x-a} + C
\]

\[
\frac{d}{dx} \left( \tan^{-1} \frac{x}{a} \right) = \frac{1}{a^2 + x^2}
\]

\[
\tan^{-1} \left( \frac{x}{a} \right) = \frac{1}{2} \ln \frac{x+a}{x-a} + C
\]

Using initial conditions \( C_0 = 0 \)

\[
2t = \tan^{-1} \left( \frac{w_2}{\sqrt{3} \, a} \right)
\]

Substituting in Eq. (12) \( \sqrt{3} \, a \)

\[
w_0 = \sqrt{3} \, a \, \text{sec} \, \theta \]

\[
w_1 = \frac{1}{2} \sqrt{3} \, a \, \text{sec} \, \theta - \text{tan} \, \theta \, w_0 = \sqrt{3} \, a \, \text{sec} \, \theta \]

\[
w_2 = \frac{1}{2} \sqrt{3} \, a \, \text{sec} \, \theta - w_0^2 = \sqrt{3} \, a \, \text{sec} \, \theta \]

A uniform rectangular lamina, the acute angle between its
diagonals is \( (2\alpha) \), the lamina is free to rotate about its Center which
is fixed; initially it started spinning about one of its diagonals with
Angular velocity \( \omega_0 = \omega \) show that it will be spinning extinct entirely
about the other diagonal after a time

\[
t = \frac{\omega}{\omega_0} \int_0^t \frac{dx}{\sqrt{1 - \omega^2 \sin^2 x}}
\]

"which is an elliptic integral"

Solved by Du Bois Reymond integration method
we also choose the axes in the following way. Ox along DC (symmetrical axes)
and therefore it is a principal axis about O.

Oy along DA. (always choose the right handed system)
Oz perpendicular to the plane of the lamina at O.

Also the moment of external forces about O is equal to 0.

Accordingly we can use Euler’s equations in the form

\[ A w_i - w_i w_j (B - C) = 0 \]
\[ B w_i - w_i w_j (C - A) = 0 \]
\[ C w_i - w_i w_j (A - B) = 0 \]

\[ DB - 2\ell, \quad OB - \ell \]
\[ A - I_x = \frac{1}{2} m \sin^2 \alpha \]
\[ B - I_y = \frac{2}{3} m \cos^2 \alpha \]
\[ C = I_z + I_y = \frac{1}{3} \rho^2 \]

\[ A, B, C = \sin^2 \alpha, \cos^2 \alpha, 1 \]

Substituting in 1, 2, 3, 4

\[ \sin^2 \alpha \cos \omega_i - w_i w_j (\cos^2 \alpha - 1) = 0 \]
\[ \sin (w_i - w_2 w_3) = 0 \] (2, 11)

\[ \cos^2 \omega_i - w_i w_j (\sin^2 \alpha - \cos^2 \alpha) = 0 \]
\[ \omega_i - w_i \omega_j = 0 \] (2, 12)

\[ \omega_i \omega_j + \cos 2\alpha \omega_2 \omega_3 = 0 \] (2, 13)

At \( \ell \) we have

\[ 5\ell = (5\ell \cos \alpha, 5\ell \sin \alpha, 0), 5\ell = \parallel \]

\[ \ell \parallel \]

The reason: the component of the A velocity in the plane of the lamina is

\[ \pm (\omega_i^2 + \omega_i^2) \] can be determined in the following way.
\[ w_1 (\omega_0 + \omega_2 \times \omega_3) = \omega_1 \omega_0 + \omega_2 \omega_3 = 0 \]

Integrating \( t \) with \( \omega_2 \), we obtain

\[ \omega_2^2 + \omega_3^2 = C = S \]

(3)

\[ C_1 = S \]

\[ \omega_2^2 + \omega_3^2 = \left[ \omega_2^2 + \omega_3^2 \right]^{\frac{1}{2}} = \omega_2 (S) \]

(2)

Eqs. (3), (5) indicate that the component of \( \omega \) velocity in the plane of the lamina \( \omega_2, \omega_3 \) remains constant at any time and is equal to \( S \).

Let \( \phi \) be the angle between this component and \( \omega_2 \), we have

\[ \tan \phi = \frac{\omega_3}{\omega_2} \]

Therefore

\[ C_2 = S \cos \phi \]

\[ \omega_3 = S \cos \phi \phi' = -2 \sin \phi \phi' \]

(6)

Substitution of (6) in (2) leads to

\[ S \sin \phi \phi'' + 2 S \cos \phi \phi' = 0 \]

(7)

Substitution of \( \phi = 0 \) initially \( \phi = 0 \)

Finally, of (6) in (2), gives

\[ \phi'' + S \cos 2 \phi \cos \phi' \sin \phi = 0 \]

(8)

\[ \phi'' = S \cos 2 \phi \cos \phi' \sin \phi \]

(9)

\[ \phi'' + \frac{1}{2} S \cos 2 \phi \sin 2 \phi = 0 \]

(10)

\[ \phi'' = \frac{1}{2} S \cos 2 \phi \sin 2 \phi \]

(11)

\[ \phi'' = -\frac{1}{2} S \cos 2 \phi \cos 2 \phi \]

(12)

\[ \phi'' = -(2 S \cos 2 \phi) \cos 2 \phi + C \]
From the initial conditions (3)
\[ \dot{\phi} = 0 \quad \Rightarrow \quad \phi = C_1 \]
\[ C_2 = \frac{\phi^2}{2} = \frac{\phi^2}{2} - \phi \cos 2\alpha \quad (11) \]

\[ \dot{\phi}^2 = \phi^2 \cos 2\alpha \left( \sin^2 \alpha - \sin^2 \phi \right) \]

From (11), of course, \( \sin^2 \phi < \sin^2 \alpha \) for \( \dot{\phi}^2 \) to be true, and \( \phi \) to be real.

For \( 0 < \alpha \leq \frac{\pi}{2} \), the limits of the motion are determined from (11) by taking \( \dot{\phi} = 0 \Rightarrow \phi = 0 \), which gives

\[ \sin^2 \phi = \sin^2 \alpha \quad \Rightarrow \quad \sin \phi = \pm \sin \alpha \]

\[ \phi = \alpha \quad \text{or} \quad \phi = -\alpha \]

at \( t = 0 \)

\[ \phi(\omega) = 0 \]

The limit \( \phi = -\alpha \) at \( \phi = \omega \)

represents the position at which the plane lamina is spinning entirely about the other diagonal \( \alpha \).

In order to find the required time, we integrate the integration of (11) in the following way.

Since \( \ddot{\phi} \) decreases with time \( t \),

\[ \ddot{\phi} = -\frac{\phi^2}{2} \cos 2\alpha \quad (11) \]

\[ \dot{\phi} \] changes from \( \alpha \) to \( -\alpha \), i.e., \( \phi \) decreases and \( \phi \) is real

integration by parts

\[ \int_0^t \frac{d\phi}{dt} \quad \text{required time} \]

\[ \int_0^\alpha \frac{d\phi}{\sqrt{\sin^2 \alpha - \sin^2 \phi}} \]

\[ t = \frac{2}{\phi} \]

\[ \int_0^\alpha \frac{d\phi}{\sqrt{\sin^2 \alpha - \sin^2 \phi}} \quad \text{Cyclic function} \]
A. Mech.

Changing the variables of integration we use the transformation

\[ \phi = u - v \quad \Rightarrow \quad \lambda = 0 \rightarrow \pi / 2 \quad (u_0 \sin \phi - \Omega u_0 \sin \alpha) \sin \alpha \]

in comparison with the the integral we have to solve now.

\[ \cos \phi \, d\phi = \sin \alpha \, d\alpha \, \cos \alpha \, d\beta \]

\[ d\phi = \frac{\sin \alpha \, d\alpha \, \cos \alpha \, d\beta}{\cos \phi} \]

since \( \cos^2 \phi - \cos^2 \alpha = \sin^2 \alpha \, (1 - \sin^2 \phi) \).

Substituting in (32), we find

\[ t = \frac{2}{\sqrt{2z}} \int_{\cos \alpha}^{\cos \phi} \sin \alpha \, d\alpha \, \cos \alpha \, d\beta \]

elliptic disc of eccentricity \( e \) and \( e^2 \).

\[ \begin{align*}
A &= I_x = \frac{1}{2} m y^2 \\
B &= I_y = \frac{3}{2} m a \\
\gamma &= \frac{2}{3} \left( 1 - e^2 \right) \cos \alpha \\
\gamma &= \frac{1}{2} \Theta \\
A' &= \frac{1}{4} m a^2 \\
B' &= \frac{1}{4} m a^2 \\
C &= A + B = \frac{2}{4} m a^2 \\
A'B'C &= 11 \times 2 \times 3
\end{align*} \]
\[ I_x = \frac{1}{2}(ny^2 - x^2) \quad \text{and} \quad I_y = \frac{1}{2} \pi y^2 x \]

\[ I_z = \frac{1}{2} M R^2 \]

\[ \begin{align*}
A &= I_x - 2 \left( \frac{3}{4} M a^2 + \frac{3}{5} M b^2 \right) \\
B &= I_y - 2 M b^2 \left( \frac{2}{3} + \frac{2}{3} \right) = \frac{3}{2} M b^2 \\
C &= I_z - 2 M b^2 \left( \frac{2}{3} M a^2 \right) = \frac{1}{2} M b^2 \left( \frac{3}{2} \right) \\
A:B:C &= \frac{1}{2} : \frac{1}{2} : \frac{1}{2} = 1:1:1:2
\end{align*} \]

Substituting in Euler equations:

1. \[ 5 \omega_i - 3 \omega_i \omega_j (B - C) = 0 \quad \rightarrow \quad \omega_j = \text{Const.2} \frac{3}{5} \sqrt{2} \]
2. \[ 5 \omega_i - 3 \omega_i \omega_j (C - A) = 0 \quad \text{because} \quad \omega_j = 0 \]
3. \[ 5 \omega_i - 3 \omega_i \omega_j (A - B) = 0 \]

From I:

\[ 5 \omega_i - 3 \frac{3}{5} \omega_i \omega_j = 0 \]

From II:

\[ 5 \omega_i + 3 \frac{3}{5} \omega_i = 0 \]

\[ \omega_i = -\frac{3 \sqrt{2}}{5} \omega_j \]

\[ \omega_i = -\frac{3 \sqrt{2}}{5} \omega_j \]

From initial conditions:

\[ \omega_i = \frac{A}{\sqrt{2}} \quad \text{and} \quad \omega_j = \frac{B}{\sqrt{2}} \]

\[ \omega_i = A \cos(2t + C) \]

From I:

\[ \omega_i = \frac{A}{\sqrt{2}} \quad \text{and} \quad \omega_j = \frac{B}{\sqrt{2}} \]

From II:

\[ \omega_i = \frac{B}{\sqrt{2}} \]
\[ \omega = 0 \Rightarrow \omega = \frac{\pm 2}{\sqrt{2}} \cos (\varphi t) \]

From I. \( \omega_2 = \frac{8\sqrt{2}}{3\pi} \omega, \quad 2 = \frac{1}{p} \left( -\frac{\sqrt{2}}{12} \right) \sin \varphi t \)

\[ \omega_2 = \frac{8\sqrt{2}}{3\pi} \sin \varphi t \]

\[ \sin \varphi t \text{ to return to rotate about the same generator} \]

\[ \omega_2 = 0 \Rightarrow \omega = \frac{2\sqrt{2}}{\pi} \rightarrow \omega = \frac{5}{4} \pi \frac{\sqrt{2}}{2} x \]

At \( t = \frac{2\pi}{p} = \frac{3\pi}{3\pi} \)

\[ \omega = \left( \frac{8\sqrt{2}}{3\pi}, 0, \frac{8\sqrt{2}}{3\pi} \right) \text{ along the same generator} \]

Another solution from I + II.

Let \( (I) + i(II) \Rightarrow \)

\[ S \left( \omega + i\omega_2 \right) + i \left( \frac{\sqrt{2}}{4} \right) \left( \omega + i\omega_2 \right) = 0 \]

\[ 5 + i5 = 0 \]

\[ 5 + i5 = 0 \]

\[ \frac{5 + i5}{5} = \frac{2}{2} \]

\[ \frac{\sqrt{2} \pi}{2} = \frac{2}{2} \pi \sqrt{2} \]

\[ \sin \frac{5}{4} \pi = \frac{5}{4} \pi \cos \gamma + i \sin \gamma \]

\[ \sin \frac{5}{4} \pi = \frac{5}{4} \pi \cos \gamma + i \sin \gamma \]
Application II

The Gyroscopic Motion

By the gyroscopic motion we mean the spinning of a symmetrical rigid body about its axis of symmetry when it is mounted, so that the direction of its angular velocity and axis change freely in space.

A typical example of such motion is the spinning top.

Example: Study the motion of a spinning symmetrical top whose apex (vertex) is fixed or at rest or on a perfectly rough plane.

Let the apex of the top be fixed as origin and let the fixed axes be the axes in space.

The motion will be studied, however, relative to a rotating frame of reference taken in the following way.

Take along the line OG, where G is the center of gravity which must exist on the symmetrical axis of otherwise density is not regular. This axis is fixed in the body.
A. Mech.

The axis $O_1$ perpendicular to
the axis $O_III$ in the meridional
plane $\{Z, Z\}$ along the direction in which
$\theta$ increases.

The axis $O_II \perp$ to the
meridional plane in the direction
in which $\phi$ increases.

It is clear that the axes
$O_II, O_III$ are not fixed in the
ace $P$, they are however rotating axes

Moreover, it is readily shown that
the three axes $O_III, O_II, O_I$ are
principal axes of inertia with the total
point $O$. [OIII is an axes of symmetry and $O_II, O_I$ are
to
planes of symmetry].

3. External Forces:

1. The force of gravity, $Mg$, along the vertical downwards;
along $OZ$ downwards. Therefore

$$Mg = MgK$$

where $K$ is a unit vector along the
rotating axes $O_I, O_II, O_III$.

2. The reaction $S$, unknown in magnitude and direction.
The Angular Velocity of the axes
\[
\dot{\theta} = \dot{\phi} + \dot{\psi}; \quad \dot{\phi} = \dot{\psi} \sin \phi + \dot{\psi} \cos \phi
\]
\[
\dot{\theta} = \dot{\phi} \sin \phi + \dot{\psi} \cos \phi
\]
\[
I, II, III \text{ are right-handed system rotating axes.}
\]

Position Velocity and Acceleration of G relative to the rotating axes.
\[
P = \omega G = \dot{\omega} + \omega \times \dot{G}
\]
\[
\dot{G} = \frac{\ddot{P}}{\omega} + \frac{\dot{\omega} \times \overrightarrow{G}}{\omega} \text{ relative to the rotating axes.}
\]
\[
\ddot{G} = \frac{\dddot{P}}{\omega} + \frac{\ddot{\omega} \times \overrightarrow{G}}{\omega} + \frac{\dot{\omega} \times (\dot{\omega} \times \overrightarrow{G})}{\omega^2}
\]

The general component of velocity in spherical polar coordinates for a point moving on the surface of a sphere.
\[
\dot{r} = \dot{\rho} \cos \phi + \dot{\phi} \rho \sin \phi
\]

Should be calculated to study the linear motion and find the reaction $S$. However the reaction $S$ can be cancelled by studying the rotational motion about the fixed point $O$.

The Rotational motion about the fixed point $O$.
\[
\dot{\omega} = \omega \times \dot{\omega} \Rightarrow \dot{\omega} + \omega \times \dot{\omega} = 0
\]
in order to study the $L$ motion about the point $O$ we use the eqn:
\[
\dot{\omega} + \omega \times \dot{\omega} = 0 \text{ (moments external forces)}
\]
but $h_0$ is the angular velocity at the top.

Therefore $h_0 = Aw_i + Bw_j + Cw_k$

Since $O_1 + O_2$ are principal axes of inertia at 0.

\[ h_0 = Aw_i + Bw_j + Cw_k \] (due to symmetry) (5)

\[ \mathbf{N}_p = (\mathbf{r} - \mathbf{r}_p) \times \mathbf{F} \quad \text{and hence} \]

\[ \mathbf{N}_0 - \mathbf{r} \times \mathbf{F} = 0 \] (due to S)

\[ + \mathbf{r} \times \mathbf{M}_g = \]

\[
\begin{bmatrix}
\dot{i} \\
\dot{j} \\
\dot{k}
\end{bmatrix} =
\begin{bmatrix}
0 & -a & c \\
a & 0 & -b \\
-c & b & 0
\end{bmatrix}
\]

Substituting from (3), (6), in (4) we have

\[
\begin{bmatrix}
\dot{i} \\
\dot{j} \\
\dot{k}
\end{bmatrix} = A \dot{w}_i + A \dot{w}_j + Cw_k
\]

\[ \mathbf{M}_g \text{ a } (\mathbf{r}_g) \]

\[ A \dot{w}_i + Cw_k \dot{\theta} = A \dot{w}_i \dot{\phi} \cos \theta = 0 \] (I)

\[ A \dot{w}_j + A \dot{w}_k \dot{\phi} \cos \theta + Cw_k \dot{\phi} \sin \theta = Mg a \sin \theta \] (II)

\[ Cw_j - A w_j \dot{\phi} \sin \theta - \dot{A} a \dot{\phi} \theta = 0 \] (III)

\[ 5 \text{ unknowns: } (\dot{w}_i, \dot{w}_j, \dot{w}_k, \dot{\theta}, \dot{\phi}) \]

Kinematic equations of motion with no forces encountered.
The Kinematical Conditions (K.C)
(i.e. Conditions without considering forces).

These Conditions can be obtained in one of the following ways:

(i) Since \( O \) is a fixed point of the top, then \( O \) the
velocity of \( G \) at a point of the body is given by
\[
\mathbf{v}_o = \mathbf{w} \times \mathbf{r}
\]
(\( \mathbf{w} \) = angular velocity of the body.)
\( G \) moves about \( O \) on a surface of a sphere with the
angular velocity \( \mathbf{w} \) of the body and radius \( \mathbf{r} \).

\[
\mathbf{v}_o = \begin{vmatrix}
\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{vmatrix} = a \mathbf{w}_1 - a \mathbf{w}_2 \quad (4)
\]

From (3), (7) \( \Rightarrow \mathbf{w}_2 = \mathbf{0} \) \( \Rightarrow \mathbf{w}_2 = \mathbf{0} \) (7) the Kinematical

(ii) Since the plane is perfectly rough (no sliding), then
\[\mathbf{v}_0 = \mathbf{v}_o = \mathbf{0}\]
\( \mathbf{v}_0 \) a point of the
top

from the law of relative velocity
\[\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{AB} \times \mathbf{r}_{AB} \]
\[\Rightarrow \mathbf{v}_A + \mathbf{v}_B = \mathbf{0} \]
\[\mathbf{w} \times (\mathbf{r}_A - \mathbf{r}_B) = \mathbf{0} \]
\[\mathbf{v}_A + \mathbf{w} \times (\mathbf{r}_A - \mathbf{r}_B) = \mathbf{0} \]
\[\mathbf{v}_{AB} = \mathbf{w} \times \mathbf{r}_{AB} \]
\[\mathbf{v}_{AB} = \mathbf{w} \times (\mathbf{r}_{A} - \mathbf{r}_{B}) \]
(iii) In order to determine the angular velocity of the top, we shall rotate the axes O'I, O'I by the Eulerian angles $\Theta$ around the axis O'I, which is fixed in the top, accordingly GI(OX') becomes OX' six (fixed in the top). O'I (OY') = O'I (fixed in the top).

Therefore $\omega = \phi + \theta - \dot{\phi} - \dot{\theta}$

$\Theta$ is the angular velocity of the axes.

$\dot{\phi}$ represents the relative angular velocity of the body (Top) relative to the axes.

$\omega = \phi \sin \theta i + \dot{\phi} j + \phi \cos \theta k + \dot{\theta} l$  \hspace{1cm} \text{(8)}$

Solution of the principal eqns $I$, $II$, $III$, $IV$, $V$

We now substitute from the kinematical conditions $\dot{\theta}$, $\dot{\phi}$, $\dot{\theta}$ $\omega_{xy} = A \phi \dot{\phi} \sin \theta + (A(-\phi \sin \theta) \dot{\theta}) - \dot{\phi}$ $\dot{\phi} \omega_{y} = \phi \omega_{y}$, i.e., $\omega_{y} = \text{Constant} = \eta$

i.e. The $\dot{\alpha}$ velocity of the top around the axis O' is symmetric.

$\omega_{y} = \phi \omega_{y}$

$\omega_{y} = \phi \cos \theta + \dot{\phi} = \text{Const} = \eta$
Every substitution of the K.C. gives or yields

\[ A ( - \Phi \sin \Theta - \Phi \theta \cos \Theta ) + c \theta - A \Phi \cos \Theta = 0 \text{(9)} \]

\[ - A ( \Phi \sin \Theta + 2 \Phi \theta \cos \Theta ) + c \theta = 0 \]

\[ \Rightarrow - A ( \Phi \sin \Theta + 2 \Phi \theta \cos \Theta ) + c \theta \sin \Theta = 0 \]

Differentiate.

\[ \Rightarrow - A \left( \frac{d}{dt} (\Phi \sin^2 \Theta) + c \theta \sin \Theta \right) = 0 \]

Integrating \( \theta \) to \( t \),

\[ - A \int d(\Phi \sin^2 \Theta) + c \theta \int \sin \Theta \, d\Theta = \text{const}. \]

\[ A \Phi \sin^2 \Theta + c \theta \cos \Theta = \text{Const.} \]

\[ A \Phi \sin^2 \Theta + c \theta \cos \Theta = H \text{ (I')} \]

[Note if the vertical position \( I' \) of \( t \) is a position of the motion, then \( H = c \Theta \).]

* Alternative method to find \( \text{(I')} \) by Angular momentum. It is clear that the moment of external forces the vertical axes \( OZ \) \( \rightarrow 0 \) (since \( Mg \) is parallel to \( OZ \) and \( S \) intersects \( OZ \) at \( 0 \)). Also, the rate of change of angular momentum = the moment of external force. Accordingly, the vertical component of the angular momentum about \( O \text{ and } H_0 = \text{Const.} \). Therefore
A. Mech.

\[ h_{20} - C L \cos \theta = A L \sin \theta = \text{const.} H \]

integrating

\[ C L \cos \theta = A L \sin \theta \] \[ \Rightarrow C L \cos \theta = A L \sin \theta = H \] \[ \Rightarrow C L \cos \theta = A L \sin \theta = H \]

H is therefore, the vertical component of h.

Also from (I) we have

\[ \dot{\phi} = \frac{C L \cos \theta}{A L} \sin \theta \text{em}^2 \theta \] \[ \text{or} \]

**Finally we substitute from the kinematical equations II, III in eqn.**

\[ \dot{\phi} = \frac{A L \sin \theta \cos \theta + C L \phi \sin \theta}{A L} \text{em}^2 \theta \text{eqn.} \]

**This can be used later to study the steady motion.**

**This eqn. (12) can be integrated by substituting for \( \phi \) from (II) taking**

\[ \dot{\theta} = \frac{d \phi}{d \theta} \text{and integrating} \theta \] \[ \text{or} \]

However, it may be much easier to use eqn. (12) instead.

According to (13) we have

\[ A \dot{\phi} = \dot{\phi} \sin \theta \cos \theta + C L \phi \sin \theta \text{em}^2 \theta \text{eqn.} \]

**Multiplying by \( \dot{\phi} \text{em}^2 \theta \) we find**

\[ A \left( \dot{\phi} \phi \sin \theta \cos \theta + \dot{\phi}^2 \phi \sin \theta \cos \theta \text{em}^2 \theta \right) \]

Substituting from (II) we find

\[ A \dot{\theta} + \frac{A L}{2} \frac{d}{d \theta} \text{em}^2 \theta \text{eqn.} \]

\[ \text{integrating} \] \[ \theta \] \[ \text{or} \]

\[ A \left( \dot{\theta} + \dot{\phi}^2 \phi \text{sin} \theta \text{cos} \theta \right) = - \text{Mg} \phi \cos \theta = \text{const.} \]

**Kinetic Energy**
Can be alternatively obtained by using the convention of energy. The only force which performs work is the Coriolis force. According to Kinetic energy, Potential energy, Constant
\[ KE = \frac{1}{2} ( C_1^2 + \dot{C}_1^2 + C_2^2) \]

Therefore,
\[ C_1 ( \dot{C}_1 \cos \theta + C_2 \cos \phi) + \frac{1}{2} ( C_1^2 + \dot{C}_1^2 + C_2^2) = MG \cos \theta \]

\[ \phi = \text{the 3 principal angles which describe the motion of a spinning body} \]

---

The Condition of steady motion:

Steady motion is defined by \( \dot{\theta} = \text{const} = \alpha \),
\[ \phi = \text{const} = \omega \]

In order to find the required conditions we substitute from (16)

\[ A \dot{\theta}^2 = \frac{1}{2} ( C_1^2 + \dot{C}_1^2 + C_2^2) \]

Either, \( \dot{\theta} = 0 \) \( \alpha = 0 \) \( \dot{\theta} = 0 \), which represents the vertical position 0.

\[ A \dot{\theta}^2 = C_1^2 \dot{C}_1 + M \cos \theta \]

Therefore, \( \dot{\theta}^2 = [C_1^2 \dot{C}_1 + M \cos \theta] / (A \alpha \cos \phi) \)

Notice (16), (17) \( \alpha \) and \( \dot{\theta} \) are real valued at the vertical position.

\( \dot{\theta} \) is real if \( C_1^2 \dot{C}_1 + M \cos \theta \geq 0 \)

which is the required condition for the steady motion is \( C_1^2 \dot{C}_1 + M \cos \theta \geq 0 \).
Note (1) If the apex, \( O \), is fixed (not at rest on a rough plane) the steady motion in the horizontal position (\( x = 3 \)) is always possible since \( c_1 x = 0 \) and (18) is satisfied.

In this position, the angular velocity \( \omega \) of the lever can be found directly from eq. (16), according to which

\[
- \dot{\theta} \dot{\theta} + M g x = 0 \quad \Rightarrow \quad \dot{\theta} = \frac{M g x}{\ddot{\theta}}
\]

Note (2) It seems from eqn. (15) that the steady motion in the vertical position (\( x = 0 \)) is possible, therefore it is of interest to study small oscillations around the vertical position \( E = 0 \).

For this motion, \( \theta \ll c \) (small quantity), \( \dot{\theta}, \ddot{\theta} = c \), as can

\[
\sin \theta = \theta - \frac{\theta^3}{6} + \text{neg. terms}
\]

From I', \( A \Phi \sin^2 \theta + C \Phi \cos \theta = H \)

Then the vertical is a position of the motion.

Substituting from (16) we have

\[
A \Phi \sin \theta + C \Phi (1 - \frac{\theta^2}{2}) = C_n
\]

and

\[
\dot{\theta} = \frac{c}{\gamma A}, \quad \ddot{\theta} = \frac{\theta}{\gamma A}
\]

which cannot be obtained from eqn. (17)

Substituting in eqn. (112), even

\[
A \dot{\theta} - A \left( \frac{c^2 \theta}{2A} \right) \sin \theta \times 1 + C \left( \frac{c^2 \theta}{2A} \right) \dot{\theta} = M g a \dot{\theta}
\]

Hence steady motion

\[
A \dot{\theta} - \frac{c^2 \theta}{4A} \theta + M g a \dot{\theta} = 0
\]

\[
\ddot{\theta} = \frac{c^2 \theta}{4A} - \frac{M g a}{4A} \theta
\]
For a stable steady motion we must have \( C_{m}^{2} > 2Mg/A \) 

subject to condition (21) eqn \((20)\) represents a SHM oscillation of periodic time \( T = 2\pi / \omega \)

\[
\frac{2\times (2A)}{\sqrt{C_{m}^{2} - 4Mg/A}} = T
\]

eqn \((21)\) equivalent to Condition \((18)\) with \( \alpha = 0 \) (vertical position) 

The limits of the steady motion: 

\[
\bar{z} = \alpha \cos \theta, \quad \bar{z}^* = -\alpha \sin \theta \n\]

in order to find the limits of the motion we substitute for \( \phi \) for eqn \((21)\) in eqn \((1)\)

\[
\phi = (H - CM \cos \theta) / A \sin^{2} \theta \quad \text{(eqn 11)}
\]

Substituting in eqn \((1)\) we find

\[
A \left[ \frac{\dot{\theta}^{2} + \sin^{2} \theta \left( \frac{H - CM \cos \theta}{A \sin \theta} \right)^{2}}{A} \right] + \frac{2MgA \cos \theta}{A} = \\
= \frac{K}{A} - 2 \left[ \frac{\dot{\theta}^{2} + \sin^{2} \theta \left( \frac{H - CM \cos \theta}{A \sin \theta} \right)^{2}}{A} \right] + \frac{2MgA \cos \theta}{A} = \frac{K}{A}
\]

\[
\dot{\theta}^{2} \sin^{2} \theta = \left( \frac{K}{A} - \frac{2MgA \cos \theta}{A} \right) \sin^{2} \theta - \left( \frac{H}{A} - \frac{CM \cos \theta}{A} \right)^{2}
\]

if \( u = \cos \theta - \frac{2}{A} \rightarrow \dot{u} = -2u \sin \theta \theta - \frac{2}{A^{2}} \sin \theta \theta \)

Substituting in eqn \((1)\)
In order to study the existence of these limits we proceed in the following way.

Let \( U^2 = H(u) = (\alpha - \beta u)(1 - u^2) - (\delta - \epsilon u)^2 \)

3rd degree eqn \( \Rightarrow \) 3 roots

- For large \( u \) dominant term \( \beta u^3 \)
- Since the dominant term for large \( u \) is \( \beta u^3 \)

\[ U = \frac{\pi}{2} \text{ one root} \]

From initial conditions:

at \( t = 0 \):

\[ U_0 \]

\[ G \rightarrow \text{one of the limits of the motion or vertical position} \]

\[ \text{the vertical position is } \frac{1}{2} \text{ position of the motion} \]
1) If \( \theta = \gamma \) the vertical position \( (\theta - v) \) then
\[
K = 2 \text{mag} \quad \alpha \neq \beta \quad (\text{double root})
\]
(which is the situation common in most problems)

Since the dominant term in \( \beta \) \( u^2 \)

\[ H^2 < KA \]

Here, there are two possible limits

For the motion.

2) If \( H^2 > KA \)

open factor

\[ \theta = \beta, \theta = \gamma \]

\[
H = cH_1, K = cK_1
\]

3) \( \beta, \gamma \)

It is clear from the figure above that if \( H^2 > KA \), the roots are possible (\( u_1, u_2 \)),

if \( H^2 < KA \), two roots are possible (\( u_1, u_2 \))

if the area \( C \) is fixed.

While there exist only one root \( u_2 \), if the area \( C \)
is at rest and perfectly rough plane. Finally from eqn. \( (26) \) we get \( \frac{t}{\mu} = \frac{a}{v} + \frac{e}{v} \) 

and this is the time.

But, if the top rolls down, \( \tau \) is the top's rotation.

\[ t = \frac{a}{\mu} \] 

\[ + \frac{e}{\mu} \] 

(Which gives an elliptic integral)

Example 1.11

A top consists of a uniform circular disc of mass \( 4 \) kg, and radius \( b \), and center \( C \) and a uniform thin rod \( AC \) of mass \( 2 \) kg and length \( 2b \) the rod is perpendicular to the plane of the disc and passes through its center \( S \).

\[ AC = 0.12 \text{ m} \] 

\[ CB = \frac{5b}{2} \] 

Initially \( AB \) is vertical, and \( B \) is fixed and the top is then given a spin

\[ 5 \sqrt{13} \text{ rad/s} \] 

about its axis.

If the top is slightly disturbed, show that its axis will roll until it will make an angle \( \theta \) with the upward vertical.

\[ \cos \theta = \frac{h}{g} \]

Perform steady motion will slightly disturbed.

1. The initial conditions.

\[ \omega_0 = \beta = 5 \sqrt{13} \text{ rad/s} \]

\[ \vartheta = \theta = 0 \text{ m/s} + a \cdot 0 \text{ m/s} \]

\[ \theta = \alpha = 0 \text{ m/s} \]
From the geometry of the LTP, let's denote: $A, C, P$

$I = C = \frac{1}{2}(4m) b^2 = 2mb^2$

$I = A = \frac{1}{2}(4m) b^2 = \frac{1}{2} m (\frac{b}{2})^2 + \frac{1}{2} m (\frac{b}{2})^2 = \frac{1}{2} m (b)^2$

$J, \text{ disc.}$

$A = mb^2 + 25mb^2 + 6mb^2 = 32mb^2$

$\mu = \frac{4m (\frac{b}{2})^2}{2mb^2} + 2mb^2 \left( \frac{b}{2} \right) = \frac{13mb}{2}$

3. The 3 Principal Axes of the LTP are:

$e_1 = \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}$

$A = \cos^2 \theta + C = \cos \theta$ $\sin \theta = H$

Therefore $H = C$ and II.

$A (\dot{e}_2^2 + \dot{q}_2^2 \sin^2 \theta) + 2 \mu g a \cos \theta = K$

From condition 2, $G = -2 \mu g a \cos \theta$ and III.

We have $\dot{q} = \frac{H - C \cos \theta}{A \sin^2 \theta}$

$C \left( \frac{1 - \cos \theta}{\sin \theta} \right)$

From condition III.

$A \dot{e}_2^2 + \dot{q}_2^2 \cos^2 \theta = \frac{1}{(A \sin^2 \theta)}$

$2 M g a \cos \theta = 2 M g a$
In order to find the limits of the motion write $\theta = \frac{\pi}{2}$, therefore,

$$\frac{c^2 \eta^2 (1 - \cos \theta)^2 - 2Mg a (1 - \cos \theta) \eta \sin \theta (1 - \cos \theta)}{A}.$$

Therefore,

$$\frac{c^2 \eta^2 (1 - \cos \theta)^2 - 2Mg a (1 - \cos \theta)^2 (1 + \cos \theta)}{A} = 0.$$

In the limits of the motion are given by

$$1 = \cos \theta \Rightarrow \theta = 0, \quad (\text{trivial root})$$

Also,

$$1 + \cos \theta = \frac{c^2 \eta^2}{2Mg a A}.$$  

$$\cos \theta = \frac{25}{16},$$

$$\theta = \cos^{-1} \frac{25}{16}.$$

A Top Contest of a uniform solid sphere of mass $5m$ and radius $b$, and of a thin uniform rod of mass $m$ and length $3b$ penetrating into the sphere to its center, the motion started with the rod vertical and the top spins given a spin $15 \sqrt{\frac{32}{5}}$. If the top is slightly disturbed show that the axis will descend until it will make an angle $\cos^{-1} (\frac{211}{11})$ with the vertical.
If slightly disturbed

\[ C^2 \frac{\pi^2}{L^2} > 4MgA \]

3. H. Oscillation, but if

\[ C^2 \frac{\pi^2}{L^2} < 4MgA \]

\[ o = \frac{(2n-1)^2}{4L^2} \]

Initial Conditions: at \( t = 0 \)

\[ \theta(t) = \alpha \theta' + \beta \sin \theta \]

\[ \theta(0) = a \theta' + c \sin \theta \]

\[ \theta(0) = \theta' + \sin \theta \]

(2) The geometry of the Top is with inertia G

\[ E = \frac{2}{5} (5m) b^2 \]

\[ A = \frac{2}{5} (5m) b^2 + 5m (3b)^2 \]

\[ A = 2m b^2 + 45m b^2 + 3mb^2 \]

\[ Ma = 5m x 3b \text{ m} x \frac{3b}{2} = \frac{33mb}{2} \]
A. Mechanics

5. Equations of Motion

The three principal equations are:

\[ \omega_3 = \frac{A}{\sqrt{A \sin^2 \theta + 2Mg a \cos \theta}} \]

\[ A \sin^3 \theta + \theta \theta_0 \cos \theta = H \quad (2) \]

\[ A \left( \theta_0^2 + \theta^2 \sin^2 \theta \right) + 2Mg a \cos \theta = k \quad (3) \]

From (1), the condition is

\[ \theta = 15^\circ \frac{\pi}{3} \]

\[ H = c \theta \quad k = 2Mg a \]

From (2) in (3),

\[ \theta_0 = \left[ \frac{c \theta (1 - \cos \theta)}{A \sin^2 \theta} \right] \]

Substituting in (2), we have

\[ A \theta_0^2 + A \sin^2 \theta \left( \frac{c^2 \theta^2}{A} \right) - 2Mg a (1 - \cos \theta) \sin^2 \theta = 0 \]

To find the limits of motion, we put \( \theta = \gamma \rightarrow \)

Therefore

\[ (c^2 \theta^2/A) (1 - \cos \gamma)^2 - 2Mg a (1 - \cos \theta) \sin^2 \theta = 0 \]

\[ \frac{c^2 \theta^2}{A} (1 - \cos \gamma)^2 - 2Mg a (1 - \cos \theta) \sin^2 \theta = 0 \]

\[ 2) \cos \theta = 1 \quad \theta = 0 \quad \text{(double rest = vertical position)} \]

\[ 1 + \cos \theta - \left( \frac{c^2 \theta^2}{A} / 2Mg a \right) \]

\[ H = c \theta \quad k = 2Mg a \]
\[ I_2 = \frac{2}{5} \left( \frac{2m}{b} \right)^2 \]

\[ I_{2,\text{hemi}} = \frac{2}{5} \left( \frac{2m}{b} \right)^2 \]

\[ \begin{align*}
M &= \text{mass of hemisphere} \\
2C &= \frac{2}{5} \left( \frac{5m}{b} \right)^2 + C - 2mb^2
\end{align*} \]

\[ \begin{align*}
A &= \frac{2}{3} \left( \frac{5m}{b} \right)^2 - 5m \left( \frac{3b^2}{8} \right)^2 + 5m \left( \frac{3b^2}{8} + 2b^2 \right)^2
\end{align*} \]

\[ A = 2mb^2 + 5m \left( \frac{3b^2}{2} + 4b^2 \right) + \frac{1}{2} mb^2 = 3mb^2 \]
\[ C = \frac{2}{3} (5m) b^2 = 2mb^2 \]

\[ D G = \frac{1}{2} \]

\[ 2G = \begin{array}{c}
\frac{2}{5} \hfill \\
\frac{2}{3} b
\end{array} \]

\[ A = \frac{2}{3} (5m) b^2 - 5m \left( \frac{1}{2} b \right)^2 \]

\[ \text{for hemisphere} \quad \|
\]

\[ \text{for the rest} \quad \|
\]

\[ 2m b^2 \]

Application (5)

The motion of a sphere on a rough plane.

(Ex 1) If the rough plane is fixed.

A uniform solid sphere of radius \( a \) is projected on a rough horizontal fixed plane of coefficient of friction \( \mu \), if the conditions at the beginnings of the motion are such that sliding (slipping) occurs initially. (Consider that the center of the sphere will describe a parabolic path for a time \( t = 2 V / g \) where \( \mu \) is the initial value of friction, show that the rolling motion will start at the center of gravity will move in a straight line with a uniform velocity as:

\[ T = \frac{3}{2} \]

\[ I = \frac{2}{3} ma^2 \]
(Sliding Motion)

Geometry and axes

The origin \( O \) is taken at the initial position of the sphere (initial position at the point of contact with the rough plane).

We then choose a frame of reference \((X, Y, Z)\) fixed in the same as along direction

\(OX\) in the rough plane into which initial
de of sliding

Since initial direction of the Velocity at the point of contact \( P \) taking \( OY \perp OX \) in the rough plane \( OZ \perp OXY \) of the plane.

Position Velocity and acceleration of the center of mass

\[
\mathbf{r} = (x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}
\]

\[
\mathbf{v} = \frac{\mathbf{\dot{r}}}{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z}) = \mathbf{v}_0
\]

\[
\mathbf{a} = \frac{\mathbf{\ddot{r}}}{\mathbf{r}} = (\ddot{x}, \ddot{y}, \ddot{z})
\]

(at \( t = 0 \), \( x = u, y = v, z = w \))

External Forces

1. \((Mg)\) acting at \( G \) vertically down
2. \(R\) Normal reaction acting at the point of contact \( P \) vertically upwards
3. The friction \( F = (F_x, F_y, 0) \) acting at \( P \) along the opposite direction of sliding (opposite direction of \( 2F_p \)), Also from the condition of sliding

\[
F = MA = \sqrt{F_x^2 + F_y^2}
\]
5. The Linear Motion:

\[
M \ddot{r} - F \text{ (external)} = \vec{r}
\]

\[
M \ddot{x} = F_x (\ddot{g}) \quad M \ddot{y} = F_y (\ddot{g}) \quad M \ddot{z} = \vec{R} - M\ddot{g}
\]

\[
M \ddot{g} = R \quad (4)
\]

6. Rotational Motion:

\[
\vec{h}_G - [N_G - \vec{h}_G + q \times M \vec{r}]^2 = \gamma
\]

\[
\vec{r} = \vec{h}_G \quad q \text{ Position Vector of G (w.r.t. R)}
\]

The rotational motion can be studied in one of the following ways:

a) Rotational motion about G

\[
\vec{h}_G - N_G
\]

b) At the point of contact P

\[
\vec{h}_G + q \times M \vec{r} - \gamma
\]

The first method yields:

\[
\vec{h}_G = (A', \vec{B}', \vec{C}')
\]

\[A', B', C' \text{ moment of inertia about the axes}
\]

\[x', y', z' \text{ passing through G parallel to } x, y, z
\]

\[\text{respectively (principal axes of inertia at } G)\]

where

\[
A' = \frac{2}{5} M a^2
\]

for a solid sphere.

\[\text{Therefore } \vec{h}_G = \frac{2}{5} M a^2 c_0\]
\[
\frac{2}{3} Ma^2 \ddot{w} - \frac{2}{3} a^2 w = (\dot{F}_p - \dot{F}_g) - F
\]

\[
\begin{vmatrix}
\dot{F}_x \\
\dot{F}_y \\
\end{vmatrix} = \begin{vmatrix}
0 \\
-\dot{a}
\end{vmatrix}
\]

\[
\begin{align*}
\dot{F}_x &= \frac{2}{3} Ma^2 \ddot{w} - \frac{2}{3} a^2 w \\
\dot{F}_y &= \frac{2}{3} Ma^2 \ddot{w} - \frac{2}{3} a^2 w
\end{align*}
\]

(b) The velocity of the point of contact,

\[
u_p = \nu_0 + \nu_p g - \nu_0 = \nu_0 + \nu_p g
\]

\[
(\dot{v}, \dot{v}, \dot{w}) + \nu_0 \times (\dot{v}_p - \dot{v}_g)
\]

\[
(\dot{v}, \dot{v}, \dot{w}) + \nu_0 \times (\dot{v}_p - \dot{v}_g) = \nu_p
\]

\[
(\dot{v}, \dot{v}, \dot{w}) + \nu_0 \times (\dot{v}_p - \dot{v}_g)
\]

\[
\ddot{v}_p = \dot{v}_p - \dot{v}_g
\]

where \( \dot{v}_p \) is the initial velocity of sliding.

Also, from the conditions of sliding.

From the condition of friction

\[
\tan \theta = \frac{F_5}{F_3} = \frac{\nu_{pl}}{\nu_{pl}} = \frac{\nu_0 + \nu_0 w_0}{\nu_0 - \nu_0 w_0}
\]
Also from eqs. (2), (3):

\( \frac{F_3}{F_x} = \frac{2}{x} \)

therefore

\( \frac{F_3}{F_x} = \frac{2}{x} = \frac{a\Omega}{a^2} \)

and

\( \frac{5 + a\Omega}{x - a\Omega} \)

Integrating with respect to time,

\( \ln \left( \frac{5 + a\Omega}{x - a\Omega} \right) = \ln \left( \frac{5 + a\Omega}{x - a\Omega} \right) + C \)

where

\( \frac{5 + a\Omega}{x - a\Omega} = C \)

which is the direction of slipping.

Therefore the direction of slipping remains constant.

The constant \( C \) in eqn. (12) is determined from the initial conditions at \( t = 0 \), \( \tan \theta = \frac{V}{x + a\Omega} = \frac{1}{\frac{V}{x + a\Omega}} \)

\( V = \frac{(U + a\Omega^2)}{U - a\Omega^2} \)

is sliding in the \( x \) direction.

Therefore the sliding remains along the \( x \) direction.

We thus conclude that, \( F_3 = 0 \) always

\( F_x = F_x \)

\( \left[ F_x^2 + F_y^2 \right]^2 = F_x^2 \) at any time.
In order to find the time of the sliding motion we calculate $t_0$ at any
time and then take $t = t_0$.

We therefore substitute from eqn (13, 14) in eqn (2), (3), (5), (6)
from eqn (2) $\frac{1}{2} x = -\mu Mg t$.

Also from eqn (3) \[ x = \frac{v_0}{a} + \frac{1}{2} at^2 \]
from eqn (5) $a_0 v_0 = 0 \Rightarrow \mu = a_0$.

Finally from eqn (16) \[ \frac{2h}{3} a_0 = -\frac{\mu Mg}{3} \] (18)

\[ a_0 = \frac{\mu}{3} Mg - \frac{5}{2} a_0 \Rightarrow a_0 = \frac{\mu}{3} Mg t + C_1 t = \frac{\mu}{3} Mg + a_0 \]

In order to find the eqn of motion during the sliding motion we substitute for $t$ from eqn (18) in (6).

Therefore \[ x = x_0 - \frac{\mu}{3} Mg \frac{v_0^2}{2} \]

\[ \frac{v_f^2}{2} \frac{\mu}{3} Mg - \frac{2uv}{\mu} y \] (19)

which, similar to page 180.

By completing the square, \[ \frac{v_f^2}{2} \frac{\mu}{3} Mg - \left( \frac{2uv}{\mu} y \right) = \frac{v_0}{\mu} + \frac{v_0^2}{2} \frac{\mu}{3} Mg \]

\[ v_f^2 = \frac{2uv}{\mu} y \]

\[ \frac{v_0^2}{2} \frac{\mu}{3} Mg \]

Also \[ V = \frac{v_0}{\mu} - \frac{1}{2} \mu Mt \]

\[ \frac{v_0^2}{2} \frac{\mu}{3} Mg \]

\[ s = \frac{1}{2} \mu Mt \]

\[ V = u + a \frac{v_0}{2} \text{ Initial velocity of sliding to find the time of sliding} \]

\[ t = \frac{v_0}{2} \frac{\mu}{3} Mg \] the required time of sliding (10).
A. Mech.

Notie for pure rolling motion: $M > \frac{k_\theta^2}{\omega^2}$

Note: The rotational motion may be studied about the point P.

In this case we have to use eqn

$$h_\theta + \frac{1}{2} M \ddot{\theta} = - (\vec{r} - \vec{r}_p) \times (-k_\theta \dot{\theta})$$

Position vector $\vec{r}$, centripetal force

$$\vec{r} \times M \ddot{\theta} = 0$$

Therefore

$$h_\theta + \frac{1}{2} M \ddot{\theta} = \frac{1}{2} M \alpha^2 = \frac{1}{2} \ddot{\theta} \quad (5)$$

$$\frac{1}{3} \alpha \omega - \gamma = 0 \quad (5')$$

$$\frac{1}{5} \alpha \omega + \ddot{\gamma} = 0 \quad (6')$$

The pure rolling motion:

In friction is sufficient to prevent sliding then pure rolling motion occurs when $\dot{y} \geq 0$, then the required conditions

$$M > \left(\frac{k_\theta^2}{\omega^2} + \alpha^2\right) \tan \alpha$$

$K_\theta$ radius of gyration, $\alpha$ = the inclination of the plane to the horizontal.

In the case of a horizontal plane, $\alpha = 0$ and the condition is always satisfied with $M > 0$.

The initial conditions for the pure rolling motion can be obtained by substituting in eqn (12) to (16):

$$\theta = \frac{1}{2} \sqrt{\frac{1}{g}} t^2$$

As regards the equations of motion these are identical

with eqn (2) to (7).
Equation (5) must be excluded.

2. The Kinematical Conditions:

\[ \dot{y} + a \omega = 0 \quad \dot{z} = a \omega \]

Also, from eqn (12), (16), and (16'), we have

\[ \dot{x} - \frac{2}{3} a \dot{\omega} = -\frac{2}{3} \dot{x} \quad (\text{from Kinematical Conditions (22), (23)}) \]

\[ \frac{7}{5} \dot{x} = 0 \quad \Rightarrow \quad \dot{x} = 0 \quad \text{or} \quad \dot{z} = 0 \quad \text{(const.)} \quad (24) \]

\[ u = \frac{3}{2} a \omega = \frac{2}{3} \dot{x} \quad (5) \]

\[ v = 2 a \omega + \frac{2}{3} \dot{z} \quad (6) \]

\[ \frac{v}{u} = \frac{2}{3} \quad (25) \]

\[ \frac{v}{u} = \frac{2}{3} \quad (26) \]

Which is the eqn of a straight line with a slope \( \frac{2}{3} \).

Note: The rotational motion can be studied about the point of contact.

Moreover, in this case (pure rolling) \( \dot{y} = 0 \).

P can be considered fixed point (instantaneously).

Let P be the instantaneous center of rotation (instantaneously fixed).

We may then write the eqns:

\[ \dot{I}_{x} = \frac{2}{3} \dot{I}_{x} \quad \dot{I}_{y} = -\frac{1}{2} \dot{I}_{y} \]

\[ \left( \begin{array}{l} \frac{2}{5} \dot{I}_{x} \\
\frac{2}{5} \dot{I}_{y} \\
\frac{2}{5} \dot{I}_{z} \end{array} \right) = 0 \]

\[ \left( \begin{array}{l} a \dot{x} \\
\dot{y} \\
\dot{z} \end{array} \right) = 0 \]
Position, velocity, and acceleration at C.
\[ \dot{\mathbf{r}} = (\dot{x}, \dot{y}) \mathbf{v} = (v_x, v_y) \]

Velocity at the point of contact
\[ \dot{\mathbf{p}} = \dot{\mathbf{v}} + \dot{\mathbf{a}} \]
\[ \dot{\mathbf{p}} = \mathbf{v} + \mathbf{a} \]

After the motion starts with sliding
\[ \dot{\mathbf{p}} = \frac{\mathbf{F}}{m} \]

3. The sliding motion

\[ \tan \theta = -\frac{y + a \omega_1}{x - a \omega_1} \]

Cons.
\[ \frac{u_p v_p}{u_p - u} = \frac{a_s \cos \alpha}{u + a_s \sin \alpha} \]

Substituting into (1)...

\[ x = u t - \frac{1}{2} a_s t^2 \cos \theta \]

\[ y = -\frac{1}{2} a_s t^2 \sin \theta + \text{const} \]

\[ a \omega_2 = \frac{5}{2} g - \frac{a_s}{2} \sin \alpha + \frac{3}{2} g \]

To find the time of sliding let \( u_p = 0 \).
\[ U_p(x) = 0 \]
\[ x = \alpha x_0 = 0 \]
\[ \eta = M g \cos \theta \]
\[ \frac{d^2x}{dt^2} = a_2 \sin \theta \]
\[ t = \frac{2 \left( x + a_2 \sin \alpha \right)}{a_2 \cos \theta} \]

\[ U_p(y) = 0 \]
\[ y + a_2 \sin \alpha = \frac{\mu g t \sin \theta + a_2 \cos \alpha - \frac{1}{2} \mu g t^2}{\frac{1}{2} \mu g \cos \theta} \]

From Initial Conditions:

\[ \tan \theta = \frac{a_2 \cos \alpha}{x + a_2 \sin \alpha} \]
\[ \frac{U_p(y)}{U_p(z)} \]
\[ \frac{V}{V_{\cos \theta}} \]
\[ \frac{a_2 \cos \alpha}{\frac{1}{2} \mu g \cos \theta} = \frac{x + a_2 \sin \theta}{\cos \theta} \]

Direction of \( U_p \) at \( t = 0 \).
A. Mech.

The eqns of motion of sliding:

\[ F_x^2 + F_y^2 = A^2 x^2 \]  (1)

\[ M_y = F_y \]  (2) ; \[ M_y + F_y = M_y \]  (3) ; \[ x + M_y = M_y \]

\[ \frac{2}{3} M_x w^2 + F_y \]  (5) ; \[ \frac{2}{3} M_x w^2 + F_y \]  (6) ; \[ \frac{2}{3} M_x w^2 + F_y \]  (7)

From (5) \[ \frac{2}{3} M_x w^2 + F_y \]  (8) ; \[ \frac{2}{3} M_x w^2 + F_y \]  (9) ; \[ \frac{2}{3} M_x w^2 + F_y \]  (10)

Therefore, energy

\[ F_x - \mu R \cos \theta = - \mu M \]

Subsitute in the 1st. of Newton's

\[ x = \mu \cos \theta \]  (11)

\[ t = \mu \sin \theta \]  (12)

\[ \frac{1}{2} a \cos \theta + \mu \frac{1}{2} \sin \theta \]  (13)

\[ a = \mu \cos \theta + \mu \sin \theta \]  (14)

In order to find the time of sliding we must put \[ V_y = 0 \]

Putting \[ V_y = 0 \]  (15) implies

\[ a = \mu \cos \theta + \mu \sin \theta \]  (16)

\[ \implies t = \frac{2}{3} \mu \cos \theta - \mu \sin \theta \]  (Time of sliding)
To get the required solution, thus using

\[ t = \frac{2V}{g} \quad \text{(limited velocity of sliding)} \]

\[ t = \frac{2V}{g} = \frac{2a s_1 \cos \alpha}{g} \]

\[ s_1 \cos \alpha = \frac{1}{2} \left( s_1 + a \Delta \right) \]

\[ \text{giving the above found answer for } t. \]

The 3rd required point is

At the end of sliding \( s_2 \) \( \Rightarrow \)

\[ \frac{2}{5} U - \frac{2}{5} a \Delta \sin \alpha \]

\[ \beta = \frac{\gamma}{\gamma} \quad \text{and } \beta \text{ direction between } \alpha \text{ at } t \to t \]

\[ \tan \beta = \frac{\gamma}{\gamma} \quad \text{or } \frac{2 a s_2 \cos \alpha}{s_1 U - 2 a s_2 \sin \alpha} \]

Initially

\[ \beta' = \tan \left( \frac{2 a s_2 \cos \alpha}{s_1 U - 2 a s_2 \sin \alpha} \right) \]

\[ \beta = \tan \left( \frac{2 a s_2 \cos \alpha}{s_1 U - 2 a s_2 \sin \alpha} \right) \]

In the 1st example (pure rolling motion part)

\[ \begin{align*}
\dot{x} &= 0 \\
\dot{y} &= 0 \\
F_x &= 0 \\
F_y &= 0 \\
M_x &= 0 \\
M_y &= 0
\end{align*} \]

\[ F_z = M_z = 0 \]

\[ \begin{align*}
F_x &= -M_x = 0 \\
F_y &= -M_y = 0
\end{align*} \]

this ensuring that the motion is indeed a pure rolling one.
1. In the note after: Pure rolling motion: 15", 60°

\[ \dot{w} + q \times \dot{\vec{e}} = 0 \quad \text{(5)} \]

From kinematical conditions, \( \dot{X} = a_w \), \( \dot{\vec{v}} = \vec{w} \times \vec{r} \quad \text{(6)} \)

Also, \((\text{5})\) gives \( X = 0 \quad \text{(7.3)} \)

Therefore, \( \text{eqn (5, 6) } \Rightarrow \dot{X} = a_w \quad \text{and} \quad \dot{\vec{v}} = \vec{w} \times \vec{r} \quad \text{(7.3)} \)

Reflections, \( \dot{\vec{v}} = \dot{\vec{e}} \quad \text{(8.7)} \)

An alternative solution for example will be:

hence eqns of motion:

\[ \ddot{X} = MG \quad \text{(8.1)} \]

\[ \ddot{\vec{e}} = F \quad \text{(8.2)} \]

which is linear motion.

(Also rotational motion about G)

\[ \dot{\vec{v}} = \dot{\vec{e}} \quad \text{(8.3)} \]

\[ \ddot{\vec{e}} = \chi + k \times \dot{\vec{e}} = 0 \quad \text{(9.4)} \]

The initial conditions causes a sliding motion at \( \ddot{X} \)

The Sliding Motion:

\[ \vec{F} = -M \vec{e} \quad \text{eqn (8.5)} \quad \vec{e} \text{ unit vector along } \vec{v} \] at time t.

hence \( \vec{e} \) is a rotating unit vector.

\[ \Rightarrow \vec{F} = -M \vec{e} \quad \text{eqn (8.5)} \]

Also the velocity of the point of contact \( \vec{v}_p = \vec{v}_a - \vec{v}_f + \dot{\vec{e}} \).

\[ \vec{v}_p = \dot{\vec{v}} + \vec{v}_f \times (\vec{r}_p - \vec{r}) = \dot{\vec{v}} + \vec{v}_f \times (-ak) \]
\[
\begin{align*}
\text{Hence } & \quad U_p = U_p e - \frac{1}{2} mg \ell^2 \quad (7) \\
\text{Differentiating } & \quad \text{with } e = 0 \quad \text{we get } e' = 0 \\
U_p & < U_p e + U_p e' - \frac{1}{2} mg \ell^2 \quad (8) \\
\text{From (4) in (8)} & \quad \text{so for } \ell \rightarrow 0 \text{ we have} \\
\ell & \rightarrow \frac{e}{k} \text{ or } (\frac{e}{k})^2 \Rightarrow \frac{e}{k} = \frac{1}{2} \left( \frac{e}{k} \right)^2 \\
\ell & \text{always in the } x \text{ plane as } k \text{ in } x \text{ plane, \quad } (e' \rightarrow 0) \\
\text{From (6) } & \quad V = \frac{1}{2} mg \ell \quad (9) \\
\text{Taking the side } & \text{conducts } e = 0 \text{ with } e = 0 \text{ we find} \\
U_p & = U_p e = \frac{1}{2} mg \ell^2 \quad (10) \\
V & = \frac{1}{2} mg \ell^2 \quad (11) \\
\text{Behind the line of sliding } & \text{but } U_p \rightarrow \infty \quad \text{as } \ell \rightarrow \infty \quad (12) \\
\text{So from } & \text{equation (13)} \quad \ell = \frac{e}{k} \\
\text{Along the } & \text{initial direction } \quad \ell = \frac{e}{k} \rightarrow \infty \\
\text{From equation (15)} & \quad \ell = \frac{e}{k} \rightarrow \infty \quad \text{or } \text{as } \ell \rightarrow \infty \\
\ell & = \frac{e}{k} \rightarrow \infty \quad \text{or } \ell = \frac{e}{k} \quad \text{as } \ell \rightarrow \infty \\
\ell & = \frac{e}{k} \rightarrow \infty \quad \text{or } \ell = \frac{e}{k} \quad \text{as } \ell \rightarrow \infty \\
\text{Rolling Motion: } & 
\end{align*}
\]
Rolling Motion: eqn (5.26) must
\[ V_p = \dot{r} + \dot{\theta} x 2 \phi \] from eqn 171 \( v_p \to 0 \)
which is condition of Rolling Motion.

Differentiating \( v_p + 2 \rightarrow \)
\[ \ddot{r} + 2\dot{r} \dot{\theta} + \ddot{\theta} x 2 = 0 \]
Subs. from \( \ddot{r} \to 0 \)
\[ \ddot{\theta} x 2 \frac{\dot{\theta}}{} x (-\frac{1}{2} \dot{\theta} x 2) = 0 \]
\[ \ddot{\theta} = \frac{1}{2} \dot{\theta} \text{ (const. } \frac{\dot{\theta}}{2}) \]
Initial condition of rolling \( \ddot{r} = \dot{\theta} x 2 \), \( t = \phi \) (12) \]

which is an \( \text{eqn 12} \) straight line.

Equation (12), \( 12 \) \( F = M \dot{\theta} = 0 \) \( \rightarrow \)
\[ F/R \rightarrow 0 < \mu \] therefore the motion is indeed pure rolling.

"Analytical Dynamics"

So far we have dealt with the problems in Dynamics by using Newton's Laws of motion, as an alternative approach, Analytical Dynamics may be utilized. This new approach (1761) has the advantage of that its results can also be applied to other branches of physics (Quantum Mechanics, Statistical Mechanics, Solid state phy... )
Generalized Coords.

The generalized coordinates of a dynamical system are the minimum number of independent parameters needed to specify the motion of the system, they are usually denoted by $(q_1, q_2, \ldots, q_n)$, where $n$ is the number of generalized coordinates.

They don't have to be measured in units of length.

**Example 1.** For a particle moving in a plane, we need two generalized coordinates which may be $x$ and $y$.

$(x, y) \in (q_1, q_2)$, $\quad \text{or} \quad (x, y) \in (q_1, q_2)$

**Example 2.** For the planar motion of a rigid body, we normally use $\mathbf{x}$ and $\mathbf{y}$, three coordinates, namely $(x, y, z)$ to describe the linear motion of the center of gravity $G$ and $\theta_1, \theta_2, \theta_3$ to determine the rotational motion about $G$.

**Example 3.** The system of forces shown in Fig. 4. is determined entirely by the three generalized coordinates $(q_1, q_2, q_3)$.

$$ \Rightarrow x = q_1, \quad x_2 = q_2, \quad x_3 = q_3$$

$$x_2 = q_1 + q_2, \quad x_4 = q_1 + q_2$$

$$x_2 = q_1 + q_2, \quad x_3 = q_1 + q_2$$

$$x_6 = q_1 + q_2 + q_3$$
Generalized Velocities and accelerations
The quantities \( (q_1, q_2, \ldots, q_n) \) are turned respectively to the generalized velocities and accelerations. They don't in general have the dimensions of velocities and accelerations.

**Holonomic and non-Holonomic Systems.**
Let \( (q_1, q_2, \ldots, q_n) \) be the \emph{generalized Coordinates} of the system and let \( (\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n) \) be the \emph{infinitesimal changes} in these coordinates in \( t \).

\( \phi(q_1, q_2, \ldots, q_n) \) are independent then the system is said to be \emph{Holonomic}, on the other hand if the infinitesimal changes \( (\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n) \) are related by some non-integrable relations (say, no relations) which result from some Constraints then the system is \emph{Non-Holonomic}.

\[
\alpha_1(q_1, q_2, \ldots, q_n) = \alpha_2(q_1, q_2, \ldots, q_n)
\]

If \( \alpha \) cannot integrate to \( \dot{q}_1 \), then be related to \( q_1 \), etc.
The number \( n - m \) is usually called the number of \emph{No. of G. Coord.} and \emph{No. of relations}.

The Holonomic system is thus characterized by the fact that the \emph{No. of Generalized Coordinates} is equal to the \emph{No. of degrees of freedom}.

**Example:** \( (q_1, q_2, \ldots, q_n) \) independent \( \Rightarrow \) \( (\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n) \) non-Holonomic system. \emph{No. of G. Coordinates} = \emph{No. of degrees of freedom}.

\( (q_1, q_2, \ldots, q_n) \) independent \( \Rightarrow \) \( (\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n) \) non-Holonomic system.

\( n - m = \emph{No. of degrees of freedom} \).
Pure rolling \( \dot{U}_p = 0 \) gives \( x = \dot{a} t + c \) and \( \dot{\theta} = \dot{a} \) for \( \theta \) not constant.

Holonomic system of two \( G \) coordinates, \( \xi, \eta \), as the system is holonomic in one generalized coordinate.

Example: Consider the motion of a sphere on a horizontally rough plane. In order to describe the motion, five generalized coordinates are needed: \( (x, y) \) to describe the linear motion about a point of gravity \( G_0 \), and the three Eulerian angles \( (\alpha, \beta, \gamma) \) to determine the angular velocity \( \mathbf{\omega} \) (rotational motion about \( G_0 \)). We can distinguish between the following two cases.

1. Sliding Motion \( \dot{U}_p = 0 \) (two), which is a constraint. The infinitesimal changes \( (\delta x, \delta y, \delta \phi, \delta \psi, \delta \theta) \) are no longer independent since they must be related due to the constraints \( \dot{U}_p = 0 \) which gives

\[
\begin{align*}
\delta x &= -\dot{a} \delta t + \delta c, \\
\delta y &= \delta \phi, \\
\delta \theta &= \delta \psi + \delta \theta + \delta \psi.
\end{align*}
\]

2. Pure rolling motion \( \dot{U}_p = 0 \) (four), is sufficient for four coordinates. The infinitesimal changes \( (\delta x, \delta y, \delta \theta, \delta \phi, \delta \psi) \) are independent and the system is holonomic. The No. of degrees of freedom equals five.

\[
\begin{align*}
\dot{x} &= \dot{a} t + \dot{c} \cos \phi + \dot{\psi} \sin \phi, \\
\dot{y} &= \dot{c} \sin \phi, \\
\dot{\theta} &= \dot{\psi} \cos \phi + \dot{\theta} + \dot{\psi} \cos \phi. \\
\end{align*}
\]
\[
\begin{align*}
\dot{x} &= a \phi \cos \phi - \alpha \cos x + \alpha \sin \phi \\
\dot{y} &= a \phi \sin \phi + \alpha \phi \\
\end{align*}
\]

are two non-integrable relations between \( \dot{x}, \dot{y}, \phi, \beta \) \( \alpha, \beta \). In other words, the system is non-holonomic, and the \( n \) of degrees of freedom is \( n = 2 \). 

Lagrange's equations for a holonomic system:

We consider a holonomic dynamical system of particles \((p_i, \dot{p}_i, \cdots)\) of masses \((m_i)\) \(i = 1, \cdots, n\) by the generalized coordinates \(q_i\) of the system. (The number \( n \) of \( q_i \) \(i = 1, \cdots, n\) is the number of degrees of freedom.)

The linear equation of motion of any particle:

\[
m_i \ddot{q}_i = \sum F_i = F_i(m_i, q_i, \dot{q}_i, \cdots)
\]

Since \((q_1, q_2, \cdots, q_n)\) are the generalized coordinates of the system, then \( \dot{q}_i = \dot{q}_i(q_1, q_2, \cdots, q_n, t) \) \((i = 1, \cdots, n)\).

\(\dot{q}_i\) determines the trajectory of the particle \(p_i\). From this eqa, we have

\[
\dot{q}_i = \frac{\ddot{q}_i}{\dot{q}_i} = \frac{\sum F_i}{\dot{q}_i} \quad (\text{with } q_i \neq 0)
\]

\(\dot{q}_i \neq 0\) implies that \( \dot{q}_i \) is a function of \( q_i, \dot{q}_i, t \) as well as \( \dot{q}_i \). \( q_i \neq 0 \) also.

The coefficient of \( \dot{q}_i \)

\[
\frac{\dot{q}_i}{q_i} = \frac{\dot{q}_i}{\dot{q}_i} \quad (\text{with } q_i \neq 0)
\]

\( \dot{q}_i\) and \( \dot{q}_i \) as well.
The next step in the derivation is to show that
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial q_i} \right) = \frac{\partial}{\partial q_i} \left( \frac{\partial T}{\partial q_i} \right) - \frac{\partial}{\partial q_i} \left( \frac{\partial T}{\partial q_i} \right) \quad (5)
\]

(The two operators \( \frac{d}{dt}, \frac{\partial}{\partial q_i} \) are not commutative.)

To prove this we note that \( \frac{\partial}{\partial q_i} \) is a \( \delta \)-function object.

Therefore
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial q_i} \right) = \sum_i \frac{\partial^2 T}{\partial q_i \partial q_i} \frac{\partial q_i}{\partial q_i} + \frac{\partial^2 T}{\partial q_i \partial q_i} \frac{\partial q_i}{\partial q_i} - \left[ \frac{\partial}{\partial q_i} \frac{\partial T}{\partial q_i} - \frac{\partial}{\partial q_i} \frac{\partial T}{\partial q_i} \right] = \frac{\partial^2 T}{\partial q_i \partial q_i}
\]

(5)

The kinetic energy of the system
\[
T = \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \quad (6)
\]

\( \dot{\mathbf{r}}_i = \dot{\mathbf{r}}_i(q, t) \Rightarrow T(q, t) = T \)

Therefore
\[
\frac{\partial T}{\partial q_j} = \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} - \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} \quad (7)
\]

Also,
\[
\frac{\partial T}{\partial q_j} - \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} \quad \text{but from (4) } \Rightarrow
\]

\[
\frac{\partial T}{\partial q_j} = \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} \quad (8)
\]

Differentiating with respect to time \( \rightarrow \)
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial q_j} \right) = \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} + \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \left( \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} \right) = \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} + \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j}
\]

(From (5) + \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} = 0)

From (11) \rightarrow
\[
= \sum_i \mathbf{F}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} + \frac{\partial \mathbf{T}}{\partial q_j}
\]

(From (11)
\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = \sum F_i \frac{\partial H}{\partial \dot{q}_s} = \text{Component of force,}
\]

\[
\frac{\partial T}{\partial q_s} = \text{Generalized component of momentum}
\]
\[ S = \int (-\dot{\phi}^2 \sin^2 \theta + \phi \cos \phi \, \dot{\phi}) \, dt \]

\[ T = \frac{1}{2} I_0 \omega^2 \text{, Kinetic energy} \]

\[ I_0 = M_0 R^2 \text{, Inertia} \]

\[ C = c, \quad N = \frac{4}{3} M_0 R^2 \]

\[ T = \frac{2}{3} M_0 R^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) \]

Potential energy \[ V = -N \, A \cos \theta \]

Lagrangian \[ L = T - V = \frac{2}{3} M_0 R^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + N \, A \cos \theta \]

which is the Lagrangian function.

In order to apply Lagrange's equations, we have to calculate \[ \frac{\partial L}{\partial \theta}, \quad \frac{\partial L}{\partial \dot{\theta}}, \quad \frac{\partial^2 L}{\partial \theta \partial \dot{\theta}} \]

\[ \frac{\partial L}{\partial \theta} = \frac{4}{3} \, M_0 R^2 \quad \frac{\partial L}{\partial \dot{\theta}} = -\frac{4}{3} \, M_0 R^2 \dot{\phi}^2 \sin \theta \cos \theta - N A \sin \theta \]

\[ \frac{\partial^2 L}{\partial \theta \partial \dot{\theta}} = 0 \]

We then substitute from eq. 14 in the Lagrangian eqn.

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \]

\[ \frac{4}{3} M_0 R^2 \dot{\phi}^2 \sin \theta \cos \theta + N A \sin \theta = 0 \]

Dividing by \( M_0 \) gives

\[ \frac{4}{3} \, \dot{\phi}^2 \sin \theta \cos \theta + \frac{N A \sin \theta}{M_0} = 0 \]

which is the 1st eqn. of motion.

Also, substituting from (3) in the Lagrangian eqn.

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = 0 \]

\[ \frac{d}{dt} \left( \frac{4}{3} M_0 R^2 \dot{\phi} \sin \theta \right) = 0 \]
\[ \phi' = \frac{2}{3} a \alpha - \frac{4}{3} \alpha \frac{d^2 \phi}{d \alpha^2} \cos \alpha + g \sin \alpha \]  

Which is the 2nd order of motion.

We substitute from \( \phi \) from \( (7) \) in \( (16) \) to get:

\[ \frac{4}{3} a \alpha - \frac{4}{3} \alpha \frac{d^2 \phi}{d \alpha^2} \cos \alpha + g \sin \alpha = 0 \]  

In order to determine the limit of motion we take:

\[ \phi = 0 \frac{d \phi}{d \alpha} \]  

in eqn \( (8) \) and integrate w.t. \( 0 \).

This, however, is not required in the Present problem.

For the steady motion, we take \( \omega^2 = \alpha \) 2 const.

\[ \dot{\alpha} = \text{const.} = \omega \]  

Substituting in eqn \( (6) \) we obtain:

\[ \omega = \frac{\sqrt{a \alpha}}{a \alpha} \sin \alpha \cos \alpha + g \sin \alpha = 0 \]

if \( \sin \alpha = 0 \) (i.e. not in the vertical position)

\[ \Rightarrow - \frac{1}{2} a \alpha^2 \sin x \cos x + g \sin x = 0 \]

The condition of steady motion \( \omega > 0 \) (real) we get:

\[ \cos x > 0 \]

i.e. \( 0 < x < \pi/2 \) \( (10) \)  

Note: Condition (11) agrees with the general condition

\[ C^2 \frac{\omega^2}{\alpha} > 4 M g a A \cos \alpha' \text{ of the steady motion of the top} \]

since in this case \( C = 0 \), and the condition takes the form:

\[ \cos x > 0, \pi/2 < x < \pi \]

where \( \alpha' \) is the angle between the axis of the top and the upward vertical \( (\alpha > \pi/2 - \alpha') \) \( (11)' \)

In order to study small oscillations about the position of steady motion, we take \( \theta = \alpha + \gamma \) \( \Rightarrow \gamma' \ll \alpha \)

Therefore, \( \dot{\gamma} = \gamma' \dot{\gamma} \) \( (12) \)
Since the position of initial steady motion \( \psi = \frac{x}{v} \) is a position of the motion then from eq. (7)
\[
\psi^2 \sin^2 \alpha = (13)
\]
Moreover, \( \sin \theta \approx \psi (1 + \psi) \approx \psi \cos \psi + \psi \sin \psi \approx \psi \cos \psi \)
For small angles i.e., \( \cos \psi \approx 1 \), \( \sin \psi \approx \psi \Rightarrow \)
\[
\sin \theta \approx \sin \psi \cos \psi = \sin \psi (1 + \psi \cot \alpha)
\]
Also, \( \cos \theta \approx \cos \psi \approx \cos \psi (1 - \psi \tan \alpha) \)

We now substitute from (12), (13), (14) a, b to obtain
\[
\frac{4}{3} a \ddot{y} - \frac{4}{3} a \psi^2 \sin^2 \alpha \cos \psi (1 - \psi \tan \alpha) =
\frac{4}{3} a \psi - \frac{4}{3} a \sin^2 \alpha (1 + \psi \cot \alpha)^2 + 8 \sin \alpha (1 + \psi \cot \alpha)
\]
\[\Rightarrow \quad a \ddot{y} = a \psi^2 \sin^2 \alpha \cos \psi (1 - \psi \tan \alpha)
\]
\[
\Rightarrow \quad \dot{\psi} = \sqrt{3} \theta / 4 \alpha \cos \alpha
\]
\[
\psi = \frac{\sin \alpha \cos \alpha (1 - \psi \tan \alpha)}{\sqrt{3} \theta / 4 \alpha \cos \alpha}
\]
\[
\sin \alpha (1 + \alpha)^n = 1 + n \alpha + \frac{n(n-1)}{2!} \alpha^2
\]
\[n = -2, \quad \psi = \frac{1}{\sqrt{3} \theta / 4 \alpha \cos \alpha} \]

\[
\Rightarrow \quad \psi - \psi^2 \sin \alpha \cos \psi (1 - \psi \tan \alpha) (1 + \frac{3}{4} \psi \cot \alpha) \Rightarrow \psi - \frac{1}{4} \psi \cot \alpha \sin \alpha \cos \psi (1 - \psi \tan \alpha - 3 \frac{4}{\sqrt{3} \theta / 4 \alpha \cos \alpha} \psi \cot \alpha)
\]
\[
\Rightarrow \quad \frac{1}{8} \psi^2 + \frac{1}{16} \psi^2 \cos^2 \alpha \cos^2 (1 + 3 \frac{4}{\sqrt{3} \theta / 4 \alpha \cos \alpha} \psi \cot \alpha)
\]
\[
\Rightarrow \quad \dot{\psi} = \frac{1}{8} \psi^2 (1 + \frac{4}{3} \cos^2 \alpha)
\]

Which is a simple harmonic oscillation about the motion of steady motion.
The angular frequency of the S.H.O oscillation is
\[ \omega = \sqrt{\frac{\theta}{2}} \]
the periodic time of the S.H.O
is
\[ T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{\theta}{2}}} = \frac{2\pi}{\sqrt{\theta}} \]

Ex.(3): A uniform rod of mass \( M \) and length \( 2a \) is free to rotate about its fixed end \( O \) a smooth ring of mass \( \gamma M \) slides along the rod and is connected to \( O \) by an elastic string of modulus \( E \) and natural length \( \ell \). Apply Lagrange's eqns to show that, when steady motion is possible \( \theta = \pi/3 \), \( \gamma = \gamma_0 - \frac{1}{\ell^2} \)
then the length of the string \( x = \frac{\sqrt{3}}{2} a \) and \( n = 1 \)

\( \frac{\text{Position of the ring}}{\text{Position in the string}} \)
\[ 2a > x > a \]

The generalized coordinates of the system are \( \theta, \varphi, x \) the axes are identical with example 11.1.

Kinetic energy of the rod: \( T \rightarrow \frac{2}{3} MA^2 (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) \)
In order to find the K.E. of the ring we note that
\[ T = (0, 0, x) \quad \dot{V} = i\dot{p} + \hat{\varphi} \times \dot{p} \]
\[ \text{angular velocity of the axes} \]
\[ \dot{p} = (x \theta, x \sin \theta \varphi, x) = 2 \dot{\varphi}. \]
\[ \text{Total K.E.} \rightarrow T = \frac{\theta}{2} MA^2 (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + \frac{1}{2} M x^2 \]
The forces which can perform work are $Mg, DMg, F$. These forces are conservative since the can be represented by potential energies.

Potential energy due to gravity $V_1 = -Mg \cos \theta - \frac{\Delta My \cos \theta}{z^2}$

The string energy $V_2 = \frac{1}{2} k (L - L_0)^2$ where $k$ = modulus of stiffness = (modulus of elasticity / 100).

$V_2 = \frac{1}{2} (nMg/a) (X-a)^2$ (5)

$V = V_1 + V_2$ (6) $\frac{(x-a)^2}{a}$

Total potential energy $V = V_1 + V_2$ (5)

which is the total potential energy of the system.

The Lagrangian function $L = T - V = M \left( \frac{2}{3} a^2 + \frac{1}{2} aX^2 \right) \dot{\theta}^2 + \frac{1}{2} \lambda M X^2 + Mg \cos \theta (a + \lambda X) - \frac{1}{2} \frac{nMg}{a} (X-a)^2$ (7)

The 1st Lagrangian eqn.

$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$ (1st)

$\frac{\partial L}{\partial \theta} = M \left( \frac{4}{3} a^2 + aX^2 \right) \dot{\theta}^2$

$\frac{\partial L}{\partial \theta} = M \left( \frac{4}{3} a^2 + aX^2 \right) \dot{\theta}^2 \sin \theta \cos \theta$

$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$

$\frac{d}{dt} \left( M \left( \frac{4}{3} a^2 + aX^2 \right) \dot{\theta} \right) = 0$

$- \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = M \left( \frac{4}{3} a^2 + aX^2 \right) \dot{\theta}^2 \sin \theta \cos \theta + Mg \sin \theta (a + \lambda X)^2 a$ (8)
The 2nd Lagrangian eqtn

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0
\]

\[
\frac{\partial L}{\partial \dot{q}} = M \left( \frac{4}{3} a^2 + \frac{1}{2} x^2 \right) \dot{\phi} \sin \theta + \frac{1}{2} \theta
\]

\[
M \left( \frac{4}{3} a^2 + \frac{1}{2} x^2 \right) \dot{\phi} \sin \theta = M \left( \frac{4}{3} a^2 + \frac{1}{2} x^2 \right) \dot{\phi} \sin \theta - M \left( \frac{4}{3} a^2 + \frac{1}{2} x^2 \right) \dot{\phi} \sin \theta
\]

\[
\left( \frac{4}{3} a^2 + \frac{1}{2} x^2 \right) \dot{\phi} \sin \theta = -\text{const} - 2 \theta
\]

the 3rd Lagrange eqtn

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = 0
\]

\[
L = -M \left( \frac{2}{3} a^2 + \frac{1}{2} \frac{x^2}{\theta} \right) \left( \dot{\phi}^2 + \dot{\phi}^2 \sin^2 \theta \right) + \frac{1}{2} \frac{M}{a} x^2
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = \frac{1}{2} \frac{M}{a} x^2 \dot{\phi} \sin^2 \theta
\]

Therefore,

\[
\frac{1}{2} \frac{M}{a} x^2 \dot{\phi} \sin^2 \theta - \frac{1}{2} \frac{M}{a} x^2 \dot{\phi} \sin^2 \theta = 0
\]

For the steady motion, \( \dot{\phi} \) constant, \( \dot{\phi} = 0 \),

Thus from (3), (3) therefore \( x \) must be a constant.

Therefore, \( \dot{\phi} = 0 \), \( x = X \), \( \dot{\phi} = 0 \) (ii)

which are the conditions of steady motion.

Substituting (ii) in eqtn. (10) we find that

\[
\left( \frac{4}{3} a^2 + \frac{1}{8} \right) \dot{\phi}^2 \sin^2 \theta + \frac{1}{2} \frac{2 a (a + 2 b) \dot{\theta}}{2} \left( \frac{a + b}{2} \right) \left( \frac{a + b}{2} \right) = 0
\]

\[
-2 \theta \dot{\phi}^2 \sin^2 \theta - \frac{1}{2} \frac{2 a (a + 2 b) \dot{\theta}}{2} \left( \frac{a + b}{2} \right) \dot{\phi}^2 \sin^2 \theta = 0
\]

eqtn (12), (13) are two eqtns in the unknowns.
\[ \delta = x, d, n; \ \text{if } \delta = \frac{139^2}{2a}, \ x = \frac{139}{2a}, \ \text{the} \]

From these two equations we find

\[ -\frac{3a^2}{2a} \cdot \frac{3}{2} + \frac{3}{2} (\delta + \frac{1}{2}) = -12 \]

\[ a - 3a^2/4a + a + \xi = 0 \rightarrow \xi = (4a/3) \]

From (13), \[-\frac{3}{2a} \cdot \frac{3}{2} - \frac{7a}{3} (\delta + \frac{1}{2}) + (n/3/a) (a/3) = 0\]

\[ \frac{3}{2} + 2n \rightarrow n = 6 \]

Generalized coordinate

Equilibrium

Function

Natural length \( a \)

Length at equilibrium \( = 3a \)

1. The generalized coordinates are \( \Theta, \phi, \psi \)

The system is conservative

2. K.E = \( T = T_{\text{K particle}} + T_{\text{K string}} = \frac{1}{2} m \xi^2 + \frac{1}{2} m \phi^2 \)

\[ \xi = \frac{3a \cos \phi}{2}, \ \phi = \frac{a^2 \sin \phi}{2} \]

As an alternative method can be calculated as in the following way.
\[ V_p - [x_p, y_p] = \left[ 2a \cos \phi + a(3 + \eta) \sin \phi, 2a \sin \phi + a(3 + \eta) \cos \phi \right] \]
\[ V_p = [x_p, y_p] = \left[ 2a \cos \phi + a(3 + \eta) \sin \phi, 2a \sin \phi + a(3 + \eta) \cos \phi \right] \]

For small oscillation about the potential equilibrium

\[ \theta, \phi, \eta, \theta, \phi, \eta \] are small quantities

Therefore,

\[ T = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} m (4a \dot{\theta}^2 + 9a^2 \dot{\phi}^2 + 12a \dot{\theta} \dot{\phi} + \dot{\eta}^2) \]
\[ = \frac{1}{2} m a^2 \dot{\theta}^2 + \frac{1}{2} \left( 4a \dot{\theta}^2 + 9a^2 \dot{\phi}^2 + 12a \dot{\theta} \dot{\phi} + \dot{\eta}^2 \right) \]

The potential energy \( V = V_1 + V_2 \) for

\[ V = (-mg \cos \theta - mg \left[ 2a \cos \phi + a(3 + \eta) \cos \phi \right]) + \frac{1}{2} k (3a + \eta - a)^2 \]
\[ = -mg \cos \theta + 3a \cos \phi + a(3 + \eta) \cos \phi + \frac{1}{2} m a^2 (2 + \eta)^2 \]

Therefore,

\[ V = mg \left[ \cos \left( 3 \theta + 3 \phi + \eta \right) \pi \right] \]

Then substituting \( \frac{1}{2} m a \left( 5 \eta^2 + 3 \phi^2 \right) + \frac{1}{2} m a \eta^2 \)

Then substituting \( L \) from

\[ \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial \dot{\theta}}{\partial \theta} \]

The two Lagrangean equs

\[ \frac{\partial L}{\partial \theta} = \frac{\partial L}{\partial \dot{\theta}} \]

will give the same results as in example 512
A. Mech.  The Theory of Small oscillations

Example: A uniform rod OA of mass (3m) and length 2a hanging about a smooth hinge at O. A particle of mass m is attached to A by an elastic string of length 3l. If the motion is restricted to the vertical plane of the rod, apply Lagrange's eqns. to study the small oscillations about the position of equilibrium. Find then the normal frequencies, the normal periods, the equivalent lengths of simple pendulum, and the normal modes vibrations.

The generalized coordinates are the two angles \( \theta \) and \( \varphi \) which the rod and the string make with the vertical downward.

The forces which can perform work are the weight \( 3mg \) and \( mg \). Therefore, the total energy is conserved. In order to apply Lagrange’s eqns., we need to determine \( T, V, L \) (cf. energy, Lagrangian).

\[
T = T_{\text{rod}} + T_{\text{particle}}
\]

\( \dot{\theta} \) is the angular velocity of the rod.

\[
T = \frac{1}{2} I_\theta \dot{\theta}^2
\]

where \( O \) is a fixed point. (planar motion)

\[
\frac{1}{2} \cdot \frac{4}{3} (3m) a^2 \dot{\theta}^2 = \frac{1}{2} (4m a^2 \dot{\theta}^2)
\]

\( T_{\text{particle}} = \frac{1}{2} m \dot{r}_p^2 \)

But \( \dot{r}_p = \dot{r}_A + \dot{r}_p - \dot{r}_A \)

\( \dot{r}_p = \dot{r}_A + 3a \dot{\theta} \cos \theta \)
\( y^2 = 4 a^2 \theta^2 + 9 a^2 \phi^2 + 2 (2a \theta ) (3a \phi) \cos (\phi - \theta) \)

\( T = \frac{1}{2} m a^2 (4 \theta^2 + 9 \phi^2 + 12 \phi \theta \cos (\phi - \theta) ) \)  

\( V = -3 m g a \cos \theta - m g a (2 a \phi + 3 a \phi \cos (\phi - \theta) ) \)

\( \text{Therefore: } L = T - V = \frac{1}{2} m a^2 \left( 8 \theta^2 + 9 \phi^2 + 12 \theta \phi \cos (\phi - \theta) \right) + m g a (5 \cos \theta + 3 \cos \phi) \)  

At the position of stable equilibrium:
\( \theta = \phi = 0 \quad \theta \) and the rod and the string are along the vertical downward.
\( \theta = \phi = 0 = \phi = 0 \quad \theta = \phi = \phi = 0 \quad (\text{Position of stable equilibrium}) \)

For small oscillations about this position, \( \theta, \phi, \dot{\theta}, \dot{\phi}, \ddot{\theta}, \ddot{\phi} \) are small quantities.

Therefore, \( \sin \theta = \theta = \frac{1}{2} \theta^2 \quad 2 \text{nd order terms only} \) \( \)  

Substituting into (5) we find that
\( L = \frac{1}{2} m a^2 \left( 8 \theta^2 + 9 \phi^2 + 12 \phi \theta \cos (\phi - \theta) \right) + m g a (5 - \theta^2/2) + 2 (3 \phi^2/2) \)

\( \frac{\text{Constant terms and can be ignored}}{1} \)

\( \Rightarrow L = \text{Constant} + \frac{1}{2} m a^2 \left( 8 \theta^2 + 9 \phi^2 + 12 \phi \theta \cos (\phi - \theta) \right) - \frac{m g a}{2} \left( 5 \theta^2 + 3 \phi^2 \right) \)  

\( \frac{\text{Lagrangian of small oscillations about the position of stable equilibrium}}{1} \)
Consequently,

\[
\frac{d^2 L}{d\theta^2} - m^2 (8 \dot{\theta}^2 + 6 \dot{\phi}) = 0
\]

\[
\frac{d^2 L}{d\phi^2} - m^2 (3 \dot{\phi} + 6 \dot{\theta}) = \frac{dL}{d\phi} - m g a \dot{\phi} = 0
\]

The 1st Lagrangian equation

\[
\frac{d}{dt} \frac{dL}{d\dot{\theta}} - \frac{dL}{d\theta} = m^2 (8 \dot{\theta} + 6 \dot{\phi}) + 5 m g a \dot{\theta} = 0
\]

\[
8 \dot{\theta} + 6 \dot{\phi} + 5 K a = 0
\]

The 2nd Lagrangian equation

\[
\frac{d}{dt} \frac{dL}{d\dot{\phi}} - \frac{dL}{d\phi} = m^2 (5 \dot{\phi} + 6 \dot{\theta}) + 3 m g a \dot{\phi} = 0
\]

\[
2 \dot{\theta} + 3 \dot{\phi} + K a = 0
\]

Solution

Seeking a periodic solution, we may take \( \theta = A \cos(wt + \phi) \) and \( \phi = B \cos(wt + \psi) \) or \( \phi = B \cos(wt + \psi) \).

Substitution of (12) into (13) gives

\[
-2 w^2 A \cos(wt + \phi) - 6 w^2 B \cos(wt + \psi) + 5 K A \cos(wt + \phi) = 0
\]

\[
(5k - 8 w^2)A - 6 W^2 B = 0
\]

\[
2 w^2 A - 3 w^2 B + K a = 0
\]

\[
-2 w^2 A + B K - 3 w^2 A = 0
\]
(13) \(\lambda_{1,2}\) are two linear equations in \(A, B\) for non-trivial solution the determinant of the coefficients must vanish, and accordingly
\[
\begin{vmatrix}
5k - 8w^2 & 6w^2 \\
-2w & k - 3w^2
\end{vmatrix} = 0
\]
\[
(5k - 8w^2)(k - 3w^2) - 12w^4 = 0
\]
\[
12w^4 - 23w^2k + 5k^2 = 0
\]
\[
\begin{align*}
12 (w^2)^2 - 23kw^2 + 5k^2 & = 0 \\
\Rightarrow (4w^2 - k)(3w^2 - 5k) & = 0
\end{align*}
\]
\[
w^2 = \frac{2k}{12} \quad 12 = 12
\]
\[
w^2 = 9/4 - 9/16; \\
w^2 = 5/13 = 5/13a^2
\]
whence normal angular frequencies
Therefore the normal angular frequencies are
\[
\omega_1, \omega_2 = \frac{1}{2} \sqrt{9/16}
\]
\[
\omega_1 = \sqrt{9/16} \Rightarrow \text{The normal frequencies}
\]
\[
\begin{align*}
\omega_1 & = \omega_1 = 2\pi \approx 3.14 \\
\omega_2 & = \omega_2 = 2\pi \approx 3.14
\end{align*}
\]
\[
\text{the normal periodic times}
\]
\[
\begin{align*}
T_1 & \approx 2\pi \sqrt{4/9} \approx 3.92 \\
T_2 & \approx 2\pi \sqrt{3/5a}
\end{align*}
\]
In order to find the equivalent lengths of simple pendulum.
we note that for a simple pendulum
Therefore, \(l_1 = 4a, l_2 = 3a/5\)
Finally, to find the normal modes of vibrations
we must take \(\omega^2 = \omega_1^2 = \omega_1^2 = \omega_1^2\) in (14)
\[ 2 \left( \frac{K}{4} \right) A_1 + B_1 \left( K - 8 \times 14 \right) = 0 \]

\[ \frac{A_1}{2} + \frac{B_1}{4} = 0 \rightarrow B_1 = 2A_1 \]

Therefore, \( \Theta = A_1 \cos (\omega t + \phi) \)

\( \Phi = 2A_1 \cos (\omega t + \phi) \)

\[
\begin{bmatrix}
\Theta \\
\phi
\end{bmatrix} = 
\begin{bmatrix}
A_1 \\
2A_1
\end{bmatrix} \cos (\omega t + \phi) \quad [1st \ mode \ of \ vibration]
\]

In this mode, the rod and the string vibrate in the same direction.

(Write to the vertical downward)

For the 2nd mode of vibration, we take:

\[ w^2 : w^2 = 5k/13 \quad \text{in either \ (1\&2)} \]

Therefore:

\[ -2 \left( \frac{5k}{3} \right) A_2 + B_2 \left( K - 5k \right) = 0 \]

\[ \frac{25}{3} A_2 + 4B_2 = 0 \rightarrow 5A_2 = -2B_2 \]

Therefore:

\[ A_2 = 6A_1 \quad B_2 = -5A_1 \]

\[ \Theta = 6A_2 \cos (\omega t + \phi) \quad \Phi = -5A_2 \cos (\omega t + \phi) \]

\[
\begin{bmatrix}
\Theta \\
\phi
\end{bmatrix} = 
\begin{bmatrix}
6 \\
-5
\end{bmatrix} \cos (\omega t + \phi) \]

which is the 2nd normal mode of vibration.

In this mode, the rod and the string vibrate in opposite directions over to the vertical downward.
The general solution of eqn.\(1\) takes the form:

\[
\theta = A_1 \cos (\omega t + \phi_1) + A_2 \sin (\omega t + \phi_2)
\]

Example: In the above example, if the string is elastic with natural length \(a\), and density \(\rho\) in the position of equilibrium. Find the normal modes of vibrations.

Note: Let \(L = \frac{1}{2} \left[ 8 \dot{\phi}^2 + 3 \phi^2 + 12 \dot{\theta} \phi (\cos(\phi - \theta)) \right] + mg \left[ 5 \cos (\phi - \theta) + 3 \cos (\phi) \right]

The general equation of motion can be obtained from eqns. \(5, 6\) in the following way:

\[
\frac{dL}{d\theta} = \frac{d}{d\theta} \left[ 8 \dot{\phi} + 6 \phi \cos (\phi - \theta) \right]
\]

\[
\frac{dL}{d\phi} = \frac{d}{d\phi} \left[ 6 \dot{\phi} \phi + 5 \sin (\phi - \theta) \right] = 5mg \sin \phi
\]

Therefore, the 1st Lagrangian equation:

\[
\frac{d}{d\theta} \left( \frac{dL}{d\theta} \right) - \frac{dL}{d\phi} \frac{d\phi}{d\theta} = 0
\]

\[
8 \ddot{\theta} + 6 \dot{\phi} \cos (\phi - \theta) - 6 \dot{\phi} \dot{\theta} \sin (\phi - \theta) + 5K \sin \phi = 0
\]

\[
8 \ddot{\theta} + 6 \dot{\phi} \cos (\phi - \theta) - 6 \dot{\phi}^2 \sin (\phi - \theta) + 5K \sin \phi = 0
\]

Which reduces to that of small oscillations.
\[ A \phi = 3 \alpha + \alpha^2 \]

\( \alpha \phi \) measures the extension from the position of equilibrium.

At the position of equilibrium,

\[ m \phi = K (3 \alpha - \alpha) \quad ; \quad K = \frac{m \phi}{2 \alpha} \]

\[ \omega^2 = \frac{K}{m} \quad \text{which is a self oscillation frequency.} \]

\[ \omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{1}{2}} \quad \text{The secular angle of time} \]

\[ \theta = A \cos (\omega t + \phi) \]

\[ \phi = \theta + \phi_0 \quad \text{initial angle of eccentricity} \]

\[ \alpha \phi = \theta + \phi_0 \]

\[ \theta = A \cos (\omega t + \phi_0) \quad \text{condition to \( (2) \)} \]

\[ \theta = A \sin \phi_0 \quad \text{condition to \( (3) \)} \]

Then, the conclusion when question may also exist with the

\[ \begin{bmatrix} \alpha \phi \\ \theta \end{bmatrix} = A \begin{bmatrix} \frac{1}{2} \sin \phi_0 \\ 1 \end{bmatrix} \cos (\omega t + \phi_0) \]

\[ A \begin{bmatrix} \frac{1}{2} \sin \phi_0 \\ 1 \end{bmatrix} \cos (\omega t + \phi_0) \]
The general equation of motion for an object moving with an initial velocity and constant acceleration is:

\[ a = \frac{v^2 - u^2}{2s} \]

where:
- \( a \) is acceleration
- \( v \) is final velocity
- \( u \) is initial velocity
- \( s \) is distance

The velocity and acceleration of the object at any time \( t \) can be found by:

\[ v(t) = u + at \]

\[ s(t) = ut + \frac{1}{2}at^2 \]

These equations can be used to determine the motion of objects in various scenarios.
The 3rd Lagrangian eqth gives
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\eta}} - \frac{\partial L}{\partial \eta} = 0 \]

\[ \frac{\partial L}{\partial \dot{\eta}} = m \ddot{\eta}, \quad \frac{\partial L}{\partial \eta} = -m g x_1 \eta \]

Therefore,
\[ m \ddot{\eta} + \frac{m g x_1}{2} \eta = 0 \]

\[ \ddot{\eta} = -\frac{k}{2} \eta \]

which is a 3 dimensional vibration of angular frequency
\[ \omega_3 = \sqrt{\frac{k}{2}} \sqrt{\frac{g}{2a}} \].

The equivalent length of simple

\[ \eta = A_3 \cos(\omega_3 t + \phi_3) \]

Also \[ \eta = \frac{x_3}{a} \cos(\omega_3 t + \phi_3) \]

The 3 normal modes of vibrations are

\[ \eta = A \begin{bmatrix} 1 \\ 6 \end{bmatrix} \cos(\omega_3 t + \phi_3) \]

\[ \eta = A_2 \begin{bmatrix} 6 \\ -5 \end{bmatrix} \cos(\omega_3 t + \phi_3) \]

Then, the general solution of the problem may also written in the

form
\[ \begin{bmatrix} \ddot{\theta} \\ \ddot{\eta} \end{bmatrix} = A_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos(\omega_1 t + \phi_1) + A_2 \begin{bmatrix} 6 \\ -5 \end{bmatrix} \cos(\omega_3 t + \phi_3) + A_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos(\omega_3 t + \phi_3) \]
Double Pendulum

\[ T = T_a + T_b = \frac{1}{2} m_1 \dot{\theta}_1^2 + \frac{1}{2} m_2 \dot{\theta}_2^2 \]

\[ \omega_B = \omega_A + \dot{\theta}_B \]

\[ T = \frac{1}{2} m_1 \dot{\theta}_1^2 + \frac{1}{2} m_2 \dot{\theta}_2^2 + m_2 g \left( \cos \theta_1 + \cos \theta_2\right) \]

\[ V = V = m g l \cos \theta_2 - m g \left( \cos \theta_1 + \cos \theta_2\right) \]

and complete as previously learnt.

\[ T = T)_{\text{hoop}} + T)_{\text{ring}} \]

\[ \frac{1}{2} I_0 \dot{\theta}_1^2 + \frac{1}{2} m_2 \dot{\theta}_2^2 \]

\[ I_0 = I_0 + 2 M a^2 \]

\[ T = \frac{1}{2} 4 m_2 \dot{\theta}_2^2 + \frac{1}{2} m_2 \left( \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} \cos(\phi - \theta) \right) - \]

\[ \frac{1}{2} m_2 \left[ 5 \dot{\phi}^2 + \dot{\phi}^2 + 2 \dot{\phi} \dot{\phi} \cos(\phi - \theta) \right] \]

\[ V = -2 m_2 g a \cos \theta - m g \left( a (\cos \theta + a \cos \phi) \right) - m g a \left( 3 \cos \theta + \cos \phi \right) \]

\[ L = T - V = \frac{1}{2} m_2 \left[ 5 \dot{\phi}^2 + \dot{\phi}^2 + 2 \dot{\phi} \dot{\phi} \cos(\phi - \theta) \right] - \frac{1}{2} m_2 \left[ 3 \dot{\theta}^2 + \dot{\phi}^2 \right] \]

Lagrange techniques etc.
The final equation becomes:

\[ 5\Theta + \Theta + 3KG = \Theta \quad \Theta = A \cos(\omega t + \phi) \]

\[ \Phi + \Theta + KQ = 0 \quad \Phi = B \cos(\omega t + \phi) \]

Eigenvalue Problem:

\[
(5K - 5\omega^2)A - \omega^2 B = 0
\]

\[
(3\omega^2 - 3KK - K) = 0
\]

\[
\omega^2 = 3K^2, \quad \omega^2 = \frac{K}{2}
\]

Thus:

\[ A = B, \quad D \]

\[ \Theta = A \cos(\omega t + \phi), \quad \Phi = A \cos(\omega t + \phi) \]

The ring is at rest with the head:

\[ x \text{ for } \omega^2 \]

\[ B_2 = 3A_2 \]

\[ \begin{bmatrix} \Theta \\ \Phi \end{bmatrix} = 2A_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cos(\omega t + \phi) \]

To prove that the center of gravity of the system is always on the vertical line:

\[ 3m \bar{x} = 2ma \sin \theta + m \left( a \sin \theta + \bar{a} \sin \phi \right) = ma \left( 3 \sin \theta + 3 \sin \phi \right) \]

For small oscillation:

\[ A \sin \theta \]

\[ m \cdot \left( 3A_2 + B_2 \right) \cos(\omega t + \phi) = 0 \]

\[ K \cdot B_2 \]
\[ T = \frac{1}{2} \sum \dot{q} \ddot{q} + \frac{1}{2} m \ddot{x} \]

\[ T = \frac{1}{2} m \ddot{x} + \frac{1}{2} m \dot{\theta}^2 + \frac{1}{2} m a^2 \left[ \dot{\phi}^2 + 2 \dot{\phi} \dot{\theta} \cos(\phi - \theta) \right] 
\]

\[ T = T_{\text{total}} = \frac{1}{2} m a^2 \dot{\theta}^2 + \frac{1}{2} m \dot{\phi}^2 + \frac{1}{2} m \phi^2 \]

\[ T = T_{\text{AB}} + T_{\text{BC}} \]

\[ T_{\text{AB}} = \frac{1}{2} m a^2 \dot{\theta}^2 - \frac{1}{2} m \dot{\phi}^2 - \frac{1}{2} m a^2 \dot{\phi}^2 - \frac{1}{2} m \dot{\phi}^2 - \frac{1}{2} m a^2 \dot{\phi}^2 = \frac{1}{2} m a^2 \dot{\theta}^2 - \frac{1}{2} m \dot{\phi}^2 - \frac{1}{2} m a^2 \dot{\phi}^2 
\]

\[ V = -mgL \sin \theta \]

\[ V = -mg \left( L \sin \theta + L \right) \]
\[ T_{k_0} = \frac{1}{2} I_0 \dot{q}^2 + \frac{1}{2} M_{k_0} \dot{z}_0^2 \]

\[ \frac{1}{2} \cos \left( \frac{1}{2} a^2 \dot{q}^2 + \frac{1}{2} \dot{z}_0^2 \cos (\dot{q}_0 + \dot{q}_0 \cos (\theta_0 - \phi)) \right) \]

\[ T = \frac{1}{2} m a^2 \left( \frac{1}{2} \dot{y}^2 + 4 \dot{q}^2 + 12 \dot{q}^2 \cos (\phi - \theta) \right) \]

\[ V = -2 \gamma y \dot{x} \dot{w} - 3 m g \left( 2 a \cos \theta + a \cos \phi \right) \]
A. Mech.  

Hamilton Equations.

The Poiss.
3) \[ \{q, \dot{q}\} = \frac{\partial \dot{q}}{\partial \dot{p}} - \frac{\partial \dot{p}}{\partial \dot{q}} = 0 \]

\[ \{\dot{p}, \dot{q}\} = \frac{\partial \dot{q}}{\partial \dot{p}} - \frac{\partial \dot{p}}{\partial \dot{q}} = 0 \]

4) If \( \vec{h} \) is the angular momentum of a particle about the fixed origin \( O \), prove that \( \{\vec{h}, \vec{r}\} = 0 \)

\[ \{\vec{h}, \vec{r}\} = \dot{p}_x \dot{r}_y - \dot{p}_y \dot{r}_x \]

\[ \{\vec{h}, \vec{r}\} = \frac{\partial \dot{r}_x}{\partial \dot{p}_y} - \frac{\partial \dot{r}_y}{\partial \dot{p}_x} \]

\[ \{\vec{h}, \vec{r}\} = \frac{\partial \dot{r}_x}{\partial \dot{p}_y} - \frac{\partial \dot{r}_y}{\partial \dot{p}_x} = 0 \]
The Hamiltonian of a holonomic conservative system:

For a holonomic conservative system of \( n \) generalized coordinates \( (q^i, \dot{q}^i, t) \) the Hamiltonian \( H \) is defined by

\[
H = L + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - \sum_{i=1}^{n} \frac{\partial L}{\partial q^i} \dot{q}^i
\]

where \( L(q^i, \dot{q}^i, t) \) is the Lagrangian function of the system. \( \dot{q}^i \) are the generalized components of momentum being defined by

\[
\dot{q}^i = \frac{\partial L}{\partial \dot{q}^i}
\]

where \( V \) is the potential function.

This which gives \( n \) linear equations in terms of \( \dot{q}^i \):

\[
H(q^i, \dot{q}^i, t) = \text{Compare } L(q^i, \dot{q}^i, t)
\]

eqt. (2) is a group of \( n \) linear equations of \( \dot{q}^i \) in terms of \( q^i \), \( (i = 1, \ldots, n) \).

(Note: that \( T \) is quadratic in \( q^i \) and therefore \( \frac{\partial T}{\partial q^i} \) is a linear in \( q^i \).)

These equations can thus be solved to find \( q^i \) in terms of \( \dot{q}^i \), \( (i = 1, \ldots, n) \).

Consequently the Hamiltonian \( H \) can be regarded as a function of \( q^i, \dot{q}^i, t \), \( (i = 1, \ldots, n) \).

\[
H = H(q^i, \dot{q}^i, t).
\]
Hamilton's Equations.

For a holonomic conservative system, Hamilton's equations are given by
\[ \dot{q}_i = \frac{\partial H}{\partial \dot{q}_i}, \quad \ddot{q}_i = \frac{\partial H}{\partial \dot{q}_i} \]

which are a set of linear differential equations.

Equation (13) is a group of \( 2N \) DE's of the 1st order.

On the other hand, Lagrange's equations are \( 2N \) DE's of the 2nd order.

Hamilton's equations can be obtained either by using Lagrange's equations or Hamilton's Principle.

+ Derivations Using Lagrange's Equations:

Since \( H = H(q_i, \dot{q}_i, t) \),

\[ dH = \sum \left( \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial H}{\partial t} dt \right) \]

Also, from (10) \( H = L(q_i, \dot{q}_i, t) + \sum \pi_i q_i \)

Therefore,

\[ dH = - \sum \left( \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \right) - \frac{\partial L}{\partial t} dt + \sum \pi_i d\dot{q}_i + \dot{q}_i d\pi_i \]

But, \( \pi_i = \frac{\partial L}{\partial \dot{q}_i} \) from (10)
And from Lagrange's Eq (2):
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad \Rightarrow \quad \frac{d}{dt} \dot{q}_i = \frac{\partial L}{\partial \dot{q}_i}
\]
Substituting in (15) we find
\[
\dot{H} = \sum (\dot{p}_i \dot{q}_i + q_i \dot{p}_i) - \frac{\partial L}{\partial t} dt + \sum (p_i \dot{q}_i + q_i \dot{p}_i)
\]
\[
\Rightarrow \dot{H} = \sum (\dot{p}_i \dot{q}_i + q_i \dot{p}_i) - \frac{\partial L}{\partial t} dt \quad \text{(4)}
\]
Comparing (4) to (4) we find that since \( \dot{q}_i \), \( \dot{p}_i \), \( dt \) are linearly independent
\[
p_i = -\frac{\partial H}{\partial q_i} \quad \Rightarrow \quad \dot{q}_i = \frac{\partial H}{\partial p_i}
\]
Also \( \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad \text{(5)} \)

Derivations by using Hamilton's Principle:
\[
S = \int^t_0 L(q, \dot{q}, t) \, dt = \int^t_0 [H(q, p, t) + \sum p_i \dot{q}_i] \, dt
\]
\[
= \int^t_0 f(q, p, \dot{q}, \dot{p}, t) \, dt \quad \text{(6)}
\]
where \( f(q, p, \dot{q}, \dot{p}, t) = \sum p_i \dot{q}_i - H(q, p, t) \quad \text{(7)} \)

In Eq (7), \( f \) is regarded as a function of \( 4n + 1 \) parameter. The \( n \) components of coordinates \( (q_1, q_2, \ldots, q_n) \), the \( n \) components of momentum \( (p_1, p_2, \ldots, p_n) \), and the \( 2n \) derivatives \( (\dot{q}_1, \dot{p}_1, \ldots, \dot{q}_n, \dot{p}_n) \), and the time \( t \)
According to Hamilton's principle, the actual path of motion is determined when $S$ is extremum. Also, according to Euler-Lagrange's Differential Principle, $S$ is extremum when

$$\frac{d}{dt} \left( \frac{\partial S}{\partial \dot{q}_i} \right) - \frac{\partial S}{\partial q_i} = 0 \quad \text{(11a)}$$

and

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{p}_i} \right) - \frac{\partial L}{\partial p_i} = 0 \quad \text{(11b)}$$

which are Hamilton's Equations.

E.g.: Consider the plane motion of a particle under the action of a central force of potential $V(r)$.

- The generalized coordinates are $(r, \theta)$.
- The kinetic energy $T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$.
- The Lagrangian $L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)$.
- The generalized components of momentum are:
  - $p_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r}$; $p_\theta = mr^2 \dot{\theta}$
  - $\dot{r} = \frac{p_r}{m}$; $\dot{\theta} = \frac{p_\theta}{mr^2}$.
The Hamiltonian $H = L + \mathbf{\mathcal{P}} \mathbf{q} = L + \mathbf{\mathcal{P}} r + \mathbf{\mathcal{P}} \dot{\theta} = L + \dot{r}^2 + m r^2 \dot{\theta}^2 = V + T = H$

Therefore, $H = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + V(r) = \frac{1}{2} m (\dot{r}^2 + r^2 / a^2) + V(r)$

From Hamilton's equations:

\[
\dot{q}_i = \frac{\partial H}{\partial \mathbf{\mathcal{P}}_i} \Rightarrow \dot{\mathbf{\mathcal{P}}}_i = \frac{\partial H}{\partial q_i} = \frac{\mathbf{\mathcal{P}}_i}{m} \quad \dot{\theta} = \frac{\partial H}{\partial \mathbf{\mathcal{P}}_\theta} = \frac{\mathbf{\mathcal{P}}_\theta}{m r^2}
\]

Also, \( \mathbf{\mathcal{P}}_r = \frac{\partial H}{\partial \mathbf{\mathcal{P}}_r} \Rightarrow \dot{\mathbf{\mathcal{P}}}_r = -\frac{\partial H}{\partial r} = -\left[ \frac{\mathbf{\mathcal{P}}_r}{m r^2} + \frac{2 V}{r} \right] \)

\( \Rightarrow m (\ddot{r} - r \ddot{\theta}^2) = -\frac{2 V}{r} \)

Also, \( \mathbf{\mathcal{P}}_\theta = \frac{\partial H}{\partial \mathbf{\mathcal{P}}_\theta} = 0 \) i.e. \( \mathbf{\mathcal{P}}_\theta = \text{Const} \)

i.e. \( m r \ddot{\theta} = \text{Const} = \hbar \)

Notes:

1. \( \frac{\partial H}{\partial t} = \sum_i \left( \frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial \mathbf{\mathcal{P}}_i} \dot{\mathbf{\mathcal{P}}}_i \right) = 0 \)

from \( \frac{\partial H}{\partial t} = 0 \)

If \( L, H \) does not depend explicitly on time, then \( \frac{\partial H}{\partial t} = 0 \) and accordingly

\[ \frac{\partial H}{\partial t} = 0 \quad H = \text{Constant.} \]
\[ \frac{dH}{dt} = \frac{\partial H}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}} + \frac{\partial H}{\partial \mathbf{p}} \cdot \dot{\mathbf{p}} \]

Note

\[ P_s = \frac{\partial T}{\partial q} \]

\[ T = \frac{1}{2} \sum \frac{m_i \dot{q}_i^2 + \dot{q}_i \cdot \mathbf{p}_i}{q_i} \]

\[ \frac{dP_s}{dt} = \sum \frac{m_i \dot{q}_i \dot{q}_i}{q_i} \]

From the proof of Lagrange's eqn.

Therefore,

\[ \sum \dot{q}_i P_s = \sum \frac{m_i \dot{q}_i^2}{q_i} - \sum \frac{m_i \dot{q}_i \dot{q}_i}{q_i} \]

But

\[ \sum \frac{\partial P_s}{\partial q_i} \dot{q}_i = \sum \frac{\partial P_s}{\partial q_i} \dot{q}_i \]

\[ \text{subject to the condition that } H \text{ does not depend explicitly on the time.} \]

\[ H = -L + \sum P_s \dot{q}_s - T + V + 2T - V + T - E_{\text{out}} \]

Since the system is conserved in energy, \( H \) is conserved.

Subject to the condition that \( H \) does not depend explicitly on the time (i.e., the system does not contain moving constraints).
A. Mech.

Note \( \mathcal{H} [q_i, p_i] = \sum (\frac{\partial \mathcal{H}}{\partial q_i} \frac{\partial q_i}{\partial p_i} - \frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial p_i}{\partial q_i}) \)

\[ \frac{d \mathcal{H}}{d p_i} = q_i \]

Also

\[ \{ p_i, \mathcal{H} \} = \sum \left( \frac{\partial \mathcal{H}}{\partial q_i} \frac{\partial q_i}{\partial p_i} - \frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial p_i}{\partial q_i} \right) - \frac{d \mathcal{H}}{d q_i} \frac{d q_i}{d p_i} \]

Consequently, Hamilton's equations take the symmetric form

\[ q_i = \{ q_i, \mathcal{H} \}, \quad p_i = \{ p_i, \mathcal{H} \} \]

(14)

Note (3) If \( f \) is any function of \( t, q_i, p_i \), then

\[ \frac{df}{dt} = \sum \left( \frac{\partial f}{\partial q_i} \frac{\partial q_i}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial p_i}{\partial q_i} \right) + \frac{df}{dt} \]

(15)

which makes it equivalent to Newtonian mechanics.
This equation is identical to the equation in Quantum Mechanics:

\[ -i \hbar \frac{\partial}{\partial t} \psi = \hat{H} \psi \]

Commutator

\[ \{ \hat{H}, \hat{\psi} \} = \frac{\hbar}{i} \frac{\partial}{\partial \xi} \]

Heisenberg

Schrödinger

From Eq. 1.5. (15) it is clear that \( \hat{H} \) is a constant of the motion and \( i \hbar \frac{\partial}{\partial t} \hat{\psi} = -\frac{\partial}{\partial \xi} \hat{H} \) and \( \hat{H} \)

\[ \{ \hat{H}, \hat{\psi} \} = 0 \] when \( \hat{H} \) does not depend explicitly on the time.

The same result can be obtained in Quantum Mechanics.

* Lagrange's and Hamilton's Equation

Function for an Electromagnetic Field.

For a holonomic system, Lagrange's equation take the form

\[ \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = F_q \]

\( F_q \) is the generalized components of forces.
Also if the system is conservative, these eq(1) may be expressed as

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \]  

\[ L = T - V; \quad \Gamma = \frac{dV}{dq_i} \]

In fact eqn (2) can still be verified if the generalized components of force are expressed in the form

\[ \Gamma = \frac{dV}{dq_i} \quad \text{and} \quad L = T - U(q_0, q_i, t) \]

Here U is called a generalized Potential Function (III, 3).

A good example of the latter case is the electromagnetic field.

* In order to show this we start by Maxwell's equation:

1. \( \nabla \times \mathbf{E} = \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \)  
   \[ \mathbf{E} = \text{electric field intensity}; \quad \mathbf{B} = \text{magnetic induction} \]
   \[ \text{M-K-S or SI}; \quad \frac{1}{c} \text{ reduces to unity}. \]

2. \( \nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \)
   \[ \mathbf{H} = \text{Magnetic field intensity}; \quad \mathbf{J} = \text{Conduction Current} \]
   \[ \mathbf{D} = \text{displacement vector} \]
   \[ \text{M-K-S or SI}; \quad \frac{4\pi}{c} \text{ reduces to unity respectively} \]
   \[ \mathbf{J} = \text{Current density} \]
   \[ \text{M-K-S}; \quad \mathbf{e} = \text{charge density} \]
   \[ \text{M-K-S or SI}; \quad \nabla \cdot \mathbf{B} = 0 \quad \text{in vacuum} \]
   \[ \mathbf{B} = \mu \mathbf{H} \quad \mu = \text{Permeability} \]
   \[ \mathbf{E} = \varepsilon \mathbf{D} \quad \varepsilon = \text{Permittivity} \]
Also, the Lorentz force is due to the EM field.

\[ \mathbf{F} = q \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \]  

(Note: In the case when \( \mathbf{B} = 0 \), from 0
\[ \nabla \times \mathbf{E} = 0 \Rightarrow \mathbf{E} = \nabla \phi \]
\[ \mathbf{F} = q \mathbf{E} = q \nabla \phi \quad \text{and the field is conservative.} \]

The above case is not applied in the general case.
(In general, we have from eq. 4 that
\[ \mathbf{B} = \text{Curl} \mathbf{A} \quad \text{since div curl is 0.}\]
A is called the magnetic vector potential.
Substituting in eq. 6, we obtain
\[ \nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} - \frac{1}{c} \frac{\partial}{\partial t} (\nabla \times \mathbf{A}) - \nabla \times \left( \frac{\partial \mathbf{A}}{\partial t} \right) \]
\[ \nabla \times (\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}) = 0 \quad \text{which implies that} \]
\[ \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = \nabla \phi \quad \text{(7)} \]
\[ \text{Since curl grad is 0.} \]

\[ \mathbf{F} = q \left( \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi + \mathbf{v} \times (\nabla \times \mathbf{A}) \right) \]  

But \( 2 \mathbf{v} \times \nabla \mathbf{A} = 2 \mathbf{v} (\nabla \times \mathbf{A}) = 2 \mathbf{v} \left( \frac{\partial \mathbf{A}}{\partial t} - \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) \)
\[
\begin{align*}
\dot{x} (\partial_r A) \frac{\partial}{\partial x} - \nabla \cdot \left( \frac{\partial A}{\partial x} \right) + \frac{1}{2} \partial_r A \frac{\partial A}{\partial y} - \nabla \cdot \left( \frac{\partial A}{\partial y} \right) &= \frac{\partial^2 A}{\partial x^2} \\
\end{align*}
\]
Equation (12) is in the same form as eq (11) (III)

The Lagrangian function \( L \) will thus be defined by

\[
L = T - U = \frac{1}{2} m u^2 - q \phi + q A \cdot u \quad (14)
\]

\[m \text{(mass of the charged particle)}\]

The generalized components of momentum will be defined by this case by

\[
P_i = \frac{\partial L}{\partial \dot{q}_i}, \quad P_x = \frac{\partial L}{\partial \dot{u}_x} = \text{m} \dot{u}_x + q A_x
\]

Therefore the momentum \( P = m u + q A \)

\( P \) is the generalized momentum

\( u \) velocity \( A \) vector potential \( L_{15} \)

The Hamiltonian function

\[
H = -\dot{L} + \sum p_j \dot{q}_j = -\dot{L} + p_1 \dot{u}_x + p_2 \dot{u}_y + p_3 \dot{u}_z
\]

\[
= -\dot{L} + m \dot{u}^2 + q A \cdot \dot{u} \quad (15)
\]

\[
\frac{1}{2} m \dot{u}^2 + q \phi \quad (15)
\]

\[
\therefore H = \frac{1}{2m} p^2 - q A \cdot \dot{u} + q \phi \quad (16)
\]